1. True or False? 1 is prime.

   Answer: False

   Solution: It isn't.

2. True or False? For all real numbers a and b such that $a < b$, $a^2 < b^2$.

   Answer: False

   Solution: As a counterexample, let $a = -9$ and $b = -8$.

3. True or False? If x, y, and z are side lengths of triangle ABC, and $x^2 + y^2 > z^2$, then the triangle ABC is acute.

   Answer: False

   Solution: Let $x = 5$, $y = 4$, and $z = 3$. Then $x^2 + y^2 > z^2$, but the triangle is right.

4. True or False? Let (a, b) and $[a, b]$ denote the greatest common divisor and least common multiple, respectively, of a and b. Then given any integers a, b, $\frac{[a,b]}{(a,b)}$ and (a, b) are relatively prime.

   Answer: False

   Solution: Let $a = 2$ and $b = 4$. Then $[a, b] = 4$, and $(a, b) = 2$, but $\frac{[a,b]}{(a,b)} = 2$.

5. True or False? $x^3 = 3$ has 3 real solutions.

   Answer: False

   Solution: It only has 1 real solution, $\sqrt[3]{3}$.

6. True or False? $12! > 12^{11}$.

   Answer: False

   Solution: $12! = 12 \times 11 \times 10 \times \cdots \times 2 \times 1$ but $12^{11} = 12 \times 12 \times \cdots \times 12$. Since 1 is an identity for multiplication we can throw it away from the left term and we're left with 11 factors which are all at most 12, therefore $12! \leq 12^{11}$.

7. True or False? If I draw some number of cards from a standard deck and don't replace them, it's more likely that I draw exactly 13 spades than that I draw exactly 14 spades.

   Answer: True

   Solution: There aren't 14 spades in a standard deck.

8. True or False? There exist two points a and b on the boundary of a circle of radius r such that the distance from a to b is r.

   Answer: True

   Solution: Let b be a 60 degrees rotation of a around r.

9. True or False? Suppose p_1 and p_2 are planes that intersect. Then for any two lines ℓ_1 in p_1 and ℓ_2 in p_2, ℓ_1 and ℓ_2 must intersect.

   Answer: False

   Solution: If p_1 and p_2 intersect, they intersect on a line or a plane. If the intersection of p_1 and p_2 is a line let ℓ_1 and ℓ_2 be parallel to that line. If the intersection of p_1 and p_2 is a plane, let ℓ_1 and ℓ_2 be parallel to each other. In either case, both lines don't intersect. Thus, the answer to this question is $\boxed{\text{False}}$.

10. True or False? 1729 can be written as the sum of two positive cubes in two different ways.

    Answer: True

    Solution: We see that $1729 = 12^3 + 1^3 = 10^3 + 9^3$.

11. Is $2^{3^{4^5}}$ even or odd?

    Answer: Even

    Solution: It's a power tower whose base is 2.

12. Is $(-1)^{100^2 \cdot 99} + 123 \cdot 456 \cdot 789 + 1 \cdot 11 \cdot 111 \cdot 1111$ even or odd?

    Answer: Even

    Solution: The first term is 1, the second term is even because of the 456, and the third term is odd, so the sum is even.

13. Suppose that x is the largest number which solves the equation $0 = x^2 - 52x + 376$ and $[x]$ is the closest integer to x. Is $[x]$ even or odd?

    Answer: Odd

    Solution:

$$\begin{aligned} 0 &= x^2 - 52x + 376 \\ 300 &= x^2 - 52 + 676 \\ 300 &= (x-26)^2. \end{aligned}$$

    Hence, $x = 26 + 10\sqrt{3} \approx 43$.

14. Compute the sum of the digits of the first five prime numbers.

    Answer: 19

    Solution: Just compute it, and make sure you remember that $11 > 10$.

15. Compute the sum $1 - 2 + 3 - 4 + \cdots - 10$.

    Answer: −5

    Solution: We have that

$$n - (n+1) = -1.$$

    We can pair up $(n, n+1)$ as $(1,2)$, $(3,4)$, $(5,6)$, $(7,8)$, $(9,10)$. We have 5 pairs, so the answer is $\boxed{-5}$.

16. Compute 106^2.

    Answer: 11236

    Solution: This is a trick with numbers ending in 5. Let us say you have a number $n5$ where n is the rest of the number. Then to compute $n5^2$ compute $n(n+1)$ and then append 25. This comes from $(10x+5)^2 = 100(x)(x+1) + 25$. After that, we simply need figure out 106. To do this simply add $105 + 106$ because $x^2 - (x-1)^2 = 2x - 1 = x(x-1)$.

17. Compute
$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32}.$$

Answer: $\mathbf{\frac{21}{32}}$

Solution: Set every denominator to 32 and sum $32 - 16 + 8 - 4 + 2 - 1 = 21$.

18. Compute $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{16\times 17}$.

Answer: $\mathbf{\frac{16}{17}}$

Solution: This sum is equal to $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \cdots - \frac{1}{17} = 1 - \frac{1}{17} = \boxed{\frac{16}{17}}$.

19. Compute the sum of the proper factors (that is, not 496 itself) of 496.

Answer: 496

Solution: 496 is a perfect number.

20. Compute $\sqrt{1^3 + 2^3 + 3^3 + \cdots + 10^3}$.

Answer: 55

Solution: Note that $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + 4 + \cdots + n)^2$, so the sum $1^3 + 2^3 + 3^3 + 4^3 + \cdots + 10^3 = (1 + 2 + 3 + \cdots + 10)^2$, and taking square roots, we obtain $\boxed{55}$.

21. How many positive integers less than 50 have exactly 4 divisors?

Answer: 15

Solution: Any third power of a prime p has 4 divisors, namely $1, p, p^2$ and p^3. Additionally, any number that is a product of two primes p and q has two divisors, namely 1, p, q and pq. So we have $6, 8, 10, 14, 15, 21, 22, 26, 27, 33, 34, 35, 38, 39, 46$ as numbers that work. There are exactly $\boxed{15}$ numbers.

22. How many ordered triples (a, b, c) are there such that $a \leq b \leq c$ and $abc = 1001$?

Answer: 5

Solution: Note that $1001 = 7 \times 11 \times 13$, so

$$\begin{aligned} 1001 &= 7 \times 11 \times 13 \\ &= 1 \times 7 \times 143 \\ &= 1 \times 13 \times 77 \\ &= 1 \times 11 \times 91 \\ &= 1 \times 1 \times 1001 \end{aligned}$$

are the $\boxed{5}$ ways to write 1001 as the product of 3 positive integers.

23. How many triangles with side length 1 tile a hexagon of side length 4?

Answer: 96

Solution: A hexagon is made up of 6 large triangles, and each large triangle can hold $7 + 5 + 3 + 1 = 16$ small ones, so the answer is $\boxed{96}$.

24. There are 2 dogs for every cat at Nathan's pet shelter. If there are 36 dogs and cats, how many dogs are there?

Answer: 24

Solution: Let d be the number of dogs and c be the number of cats. Then $d + c = 36$ and $2c = d$, so $d = \boxed{24}$.

25. Victoria can paint her house in 2 hours, and Clark can paint his house in 6 hours. If the area to be painted on Clark's house is twice as big as the area to be painted on Victoria's, how many hours would Victoria and Clark take to paint Victoria's house together?

Answer: $\mathbf{\frac{6}{5}}$

Solution: Clark would paint Victoria's house in 3 hours, and working together, they would take

$$\frac{1}{\frac{1}{2}+\frac{1}{3}} = \boxed{\frac{6}{5}}$$

hours.

26. How many positive integers n satisfy $n^{\frac{1}{n}} = \sqrt{2}$?

Answer: 2

Solution: Raise both sides to the power of $2n$ to get $n^2 = 2^n$. Graphing $y = x^2$ and $y = 2^x$ shows that they intersect at two positive values of x, which are $x = 2$ and $x = 4$. This gives $\boxed{2}$ solutions.

27. Let $ABCD$ be a square and let M be the midpoint of CD. How many points X are there on the boundary of $ABCD$ such that the area of AMX is one quarter of the area of $ABCD$?

Answer: 3

Solution: The set of points X such that AMX has area one quarter of the area of $ABCD$ is two lines. One of these lines intersects the square at D, and the other intersects the square at C and the midpoint of AB.

28. How many ordered pairs (a, b) of integers are there with $a^2 + b^2 = 25$?

Answer: 12

Solution: $25 = 3^2 + 4^2 = 5^2 + 0^2$, and we have to add the negative versions of those, but be careful that $-0 = 0$. This gives us 6 ways for unordered pairs. We multiply by 2 to account for the ordered pairs to obtain $\boxed{12}$.

29. Three distinct circles are drawn on the plane. There are n points that lie on at least 2 circles. How many possible values are there for n?

Answer: 7

Solution: 0 points: Just have the circles not intersect
1 points: One circle does not intersect and two circles are tangent
2 points: One circle does not intersect and two circles intersect at 2 points
3 points: Two circles except one of them is tangent with one of the two other circles
4 points: Same as above except instead of tangent, we intersect at 2 points
5 points: Have the three circles intersect at one point
6 points: That's just a Venn diagram.
That's $\boxed{7}$ numbers.

30. What is the largest perfect square that divides $7^6 - 1$?

Answer: 144

Solution 1: We have that $7^6 - 1 = (7^5 + 7^4 + 7^3 + 7^2 + 7 + 1) \cdot 6 = 6 \cdot (57 + 7^3 + 7^4 + 7^5) = 6 \cdot (57 + 7^3(1 + 7 + 7^2)) = 6 \cdot (57 + 7^3 \cdot 57) = 6 \cdot 57 \cdot (1 + 7^3) = 6 \cdot 57 \cdot 344 = 6 \cdot 3 \cdot 19 \cdot 8 \cdot 43 = 12^2 \cdot 19 \cdot 43$. Hence, the answer is $\boxed{144}$.

Solution 2: $7^6 - 1 = (7^3 - 1)(7^3 + 1) = 342 \cdot 344 = 9 \cdot 2 \cdot 19 \cdot 4 \cdot 2 \cdot 43 = 12^2 \cdot 19 \cdot 43$.

31. What is the leftmost digit of $\sqrt{99999}$?

Answer: 3

Solution: This number is slightly less than 100 times $\sqrt{10}$, so the leftmost digit is $\boxed{3}$.

32. Let (x, y) be an integer solution to

$$x^2 - 163y^2 = 1$$

such that $|x + y|$ is minimal. What is $|x + y|$?

Answer: 1

Solution: Note that $(1, 0)$ is a solution. The only way we can get something smaller is if $x = -y$. Suppose $x = -y$. Let $x^2 = y^2 = n$. Thus, if

$$x^2 - 163y^2 = 1$$

then

$$-162n = 1$$

But 1 is not divisible by 162, a contradiction. Hence there are no solutions with $x = -y$, so $(1, 0)$ is the solution with $|x + y|$ minimal. Calculating it, we have $|1 + 0| = \boxed{1}$.

33. Determine the units digit of $1^{10} + 10^9 + 110^8 + 1110^7$.

Answer: 1

Solution: Any power of a multiple 10 has units digit 0, so the only number that contributes to the unit digit is the first one, giving us $1^{10} = \boxed{1}$.

34. Find the area between the lines $y = 0$, $x = 2$, and $y = 3x$.

Answer: 6

Solution: This is a triangle of width 2 and height $3 \cdot 2 = 6$. So the area is $\boxed{6}$.

35. What is the slope of the line tangent to the curve $y = 1 - x^2$, when $x = 0$?

Answer: 0

Solution: Just graph the parabola, and its tangent at its maximum point is $\boxed{0}$.

36. If I travel around the unit circle 16 times, how far have I traveled?

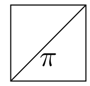
Answer: 32π

Solution: Circumference of the unit circle is 2π and we multiply it by 16.

37. A square has a diagonal of length π. Calculate the area of the square, in terms of π.

Answer: $\pi^2/2$

Solution:



Since π is the length of the diagonal, we can calculate the side length of the triangle using the Pythagorean Theorem, giving us $x^2 + x^2 = \pi^2$. From here, we see that $x^2 = \boxed{\pi^2/2}$ which is our desired area.

38. Sanat, Ankit, and Moor are throwing a party. Each of them invites some number of guests, but because of poor planning, some guests may have been invited multiple times. Sanat invites 7 guests, Moor invites 12 guests and Ankit invites 6 guests. Compute the sum of all possible numbers of total invited guests.

    Answer: 259

    Solution: The number of invited guests could be anywhere from 12 to 25 guests. Summing these up, we have $12 + 13 + \cdots + 25 = \boxed{259}$.

39. If

$$\begin{aligned} x + y - z &= 42 \\ x - y + z &= 65 \\ -x + y + z &= 12, \end{aligned}$$

    what is $x + y + z$?

    Answer: 119

    Solution: Adding the three equations, we have $x + y + z = 42 + 65 + 12 = \boxed{119}$.

40. Debbie is buying pineapples, durians, and candy. Pineapples cost 3 dollars apiece, durians 4 dollars apiece, and candies cost 1 dollar apiece. Debbie has 15 dollars to spend and she must spend all her money. What is the minimum number of items she can buy?

    Answer: 4

    Solution: A heuristic argument gives us a greedy argument, where we buy as many of the items that cost the most, then we buy the items that cost the second most, etc. This gives us 3 durians and 1 pineapple. Thus, we get $\boxed{4}$ items.

41. Some freshmen are building a zipline from their dorm and are testing the strength of the zipline with weights. Every time the rope holds, they add an additional weight to the zipline with a quarter of the weight of the previous weight that they added. They start with a 100 lb weight and the zipline breaks after they have added 200 weights. To the nearest integer, how much weight can the zipline support, in pounds?

    Answer: 133

    Solution: This is a geometric series of ratio 1/4 and initial value 100. Therefore we can plug it in: it's

$$\frac{100}{1 - \frac{3}{4}} = \frac{400}{3} = 133.333\cdots \approx \boxed{133}.$$

42. The Jupiterian currency consists of \$1 and \$12 bills, and I only have \$12 bills. I'm at a restaurant ordering pot stickers that cost \$10 each. If I don't want any change, what's the minimum number of pot stickers I can order?

Answer: 6

Solution: We need the least common multiples of the two factors 12 and 10. This is 60, and dividing by 10, since they're \$10 each, we get $\boxed{6}$.

43. If n is an irreducible whole number such that $n+2$, n^2+2, n^3+2, n^4+2, and n^5+2 are all prime, what is n? (An irreducible whole number is a number n that is not reducible, that is it cannot be written as a product pq, where p and q are integers and neither p nor q are ± 1).

Answer: 1

Solution: if $n = 1$, then all of these numbers are 3, which is prime. Otherwise, if $n \neq 3$, then $n^2 + 2$ is divisible by 3. If $n = 3$, we have $n^5 + 2 = 245$, which is not prime. Hence, $n = \boxed{1}$ is the only solution.

44. You stand on flat ground and throw an orange upwards. The orange travels along the trajectory of $y = 14t - 4.9t^2$, where t is the time in seconds after you release the orange. What is the orange's maximum height?

Answer: 10

Solution: We complete the square by $y - 10 = -10 + 14t - 4.9t^2$, so $y - 10 = -\frac{1}{10}(7x - 10)^2$. So the maximum height happens when $7x = 10$, and it is $\boxed{10}$ metres.

45. Cailan reserves rooms for BmMT problem writing sessions, which takes place from 6:30 to 8:30, but there's a $\frac{2}{3}$ chance that he makes a mistake and reserves the room until 7:30. If he makes a mistake, there's a $\frac{1}{2}$ chance that the problem writers will be allowed to stay until 8:30 anyways. What is the probability that the problem writers can stay in the reserved room until 8:30 for the next two meetings?

Answer: $\mathbf{\frac{4}{9}}$

Solution: If Cailan reserves the room correctly, the chance the problem writers are allowed to stay is 1. Thus the chance Cailan does not make a mistake and the problem writers are allowed to stay is $\frac{1}{3} \cdot 1 = \frac{1}{3}$.

If Cailan does not reserve the room correctly, the chance the problem writers are allowed to stay is $\frac{1}{2}$. Thus the chance Cailan makes a mistake but the problem writers are still allowed to stay is $\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$.

For each meeting, then, the overall probability they are allowed to stay is $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. Therefore, the chance they can stay for both meetings is $\frac{2}{3} \cdot \frac{2}{3} = \boxed{\frac{4}{9}}$.

46. Let a_i be a sequence defined as follows:

(i) $a_1 = 5$.

(ii) a_{i+1} is the front digit of the value $\overline{a_1 a_2 a_3 \dots a_i}^2$, where $\overline{abc}$ is the concatenation of digits a, b, c. For example, if $a = 1$, $b = 2$, $c = 3$, then $\overline{abc}$ is 123.

What is the value of $a_1 + a_2 + a_3 + a_4 + \cdots + a_{2017}$?

Answer: 4037

Solution: Calculating a few values, one obtains: $a_1 = 5$, $a_2 = 2$, $a_3 = 2$, ... So we might be tempted to make the conjecture $a_i = 2$ for each $i > 1$. Indeed, this is correct:

Base case was proven for $i = 2$.

Induction step: Let $52\ldots2 = b_n$ where there are n 2s. Then $b_{n+1}^2 = (b_n \times 10 + 2)^2 = 100b_n^2 + 40b_n + 4$. The sum "carries" at most $n+1$ times, but $100b_n^2$ has at least $n+2$ digits. And because the frontmost digit of b_n^2 is 2 (by our induction hypothesis), the frontmost digit of b_{n+1}^2 is 2 as well. Thus, we have

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_{2017} = 5 + 2(2016) = \boxed{4037}$$

47. Define a recursive function $F_{n+1} = (F_n)^2 F_{n-1} + F_{n-1}$. If $F_0 = F_1 = 0$, determine F_{2017}.

    Answer: 0

    Solution: Note that F_n is a multiple of F_{n-2}. Since we start at 0 for the first two, we see that all of the values are 0.

48. Let x be the sum of positive integers less than or equal to 2017 that do not share any common factors greater than 1 with 2017. Compute $\frac{x}{2017}$. Express your answer as an integer or common fraction in lowest terms.

    Answer: 1008

    Solution: 2017 is prime, so all numbers less than 2017 do not share any common factors with 2017. So we have the sum $1 + 2 + 3 + 4....2016 = \frac{2016 \cdot 2017}{2} = 2017 \cdot 1008$. Dividing this by 2017 we have 1008, our answer.

49. 2017 helicopters consume 2017 gallons of fuel in total over a span of 2017 seconds in UC Berkeley's air show. How much fuel does one helicopter consume over a span of a minute? Express your answer as a common fraction in lowest terms.

    Answer: $\frac{60}{2017}$

    Solution: If 2017 helicopters consume 2017 gallons of fuel over a span of 2017 seconds, then 1 helicopter consumes 1 gallon of fuel over a span of 2017 seconds. So over 1 second, a helicopter consumes $\frac{1}{2017}$ gallons of fuel. Therefore, over a minute, one helicopter consumes $\frac{60}{2017}$ gallons of fuel.

50. Alice has 2017 cards, labeled 1 through 2017. At first, the cards are stacked in order, with the 1 card on top. Alice chooses a random prime number p less than or equal to the number of cards in the deck, and removes the pth card from the top. Alice repeats this process 2016 times. What is the number on the card remaining?

    Answer: 1

    Solution: The top card can never be removed from the deck, and always stays the top card. Thus, the 1 card is the remaining card.