1. Nikhil has 20 BMT shirts and 24 BmMT shirts. Wen takes half of his shirts, and then Arjun takes half of his remaining shirts. How many shirts does Nikhil have left?

   Answer: 11

   Solution: Nikhil has a total of $20 + 24 = 44$ shirts before Wen and Arjun take any. After Wen takes half of them, there are $\frac{1}{2} \times 44 = 22$ shirts left, and after Arjun takes half of the 22 remaining shirts, there are $\frac{1}{2} \times 22 = \boxed{11}$ shirts left.

2. What is the value of $2 + \frac{1}{2+\frac{1}{4}}$?

   Answer: $\mathbf{\frac{22}{9}}$

   Solution: We compute:

$$2 + \frac{1}{2 + \frac{1}{4}} = 2 + \frac{1}{\frac{9}{4}} = 2 + \frac{4}{9} = \boxed{\frac{22}{9}}.$$

3. Rohit has a garden in the shape of a square with perimeter 2024. Youngmin has a lawn in the shape of a rectangle whose width is $\frac{1}{23}$ the width of Rohit's garden and whose length is $\frac{1}{22}$ the length of Rohit's garden. What is the area of Youngmin's lawn?

   Answer: 506

   Solution: Since the perimeter of Rohit's garden is a square with perimeter 2024, each of the four sides must have length $\frac{2024}{4} = 506$. Notice that $22 \times 23 = 506$, meaning that the width of the Youngmin's lawn is $\frac{506}{23} = 22$ and the length of Youngmin's lawn is $\frac{506}{22} = 23$. Thus, the area of Youngmin's lawn is $22 \times 23 = \boxed{506}$.

4. Anthony, Boris, and Carson are playing a game of frisbee! Anthony starts with the frisbee. On each turn, the player holding the frisbee throws it to someone else, after which the frisbee is either caught or dropped. If the frisbee is dropped, the nearest player picks it up. Throughout the game, Anthony throws the frisbee 8 times and catches it 6 times, Boris throws the frisbee 10 times and catches it 7 times, and Carson throws the frisbee 7 times and catches it 8 times. How many times is the frisbee dropped during the game?

   Answer: 4

   Solution: Each catch is the result of a throw, and the remaining throws must have been dropped. So, the total number of drops is the difference between the total number of throws and the number of catches, which is $(8 + 10 + 7) - (6 + 7 + 8) = 25 - 21 = \boxed{4}$.

5. Find the largest integer less than 2024 that leaves a remainder of 20 when divided by 24.

   Answer: 2012

   Solution 1: We first solve for the largest integer x such that $24x+20 \leq 2024$ because our answer is then $24x + 20$. Isolating the variable gives $24x \leq 2004$, which means $x \leq \frac{2004}{24} = \frac{167}{2} = 83.5$. Thus, $x = 83$ because x is an integer, which means our answer is $24(83) + 20 = \boxed{2012}$.

   Solution 2: Upon dividing 2024 by 24, we get a remainder of 8. However, we want our remainder to be $20 = 8 + 12$, which means our desired integer is $2024 + 12 - 24 = \boxed{2012}$.

6. Find the sum of all prime numbers less than 50 that have only prime digits.

   Answer: 77

   Solution: We know that the single digit primes are 2, 3, 5, and 7.

For double-digit prime numbers consisting of only prime digits, the units digit cannot be 2 because otherwise the number would be divisible by 2, and it cannot be 5 because otherwise the number would be divisible by 5. Thus, we know the units digit must be either 3 or 7. Knowing this, we have two candidates for units digit 3: 23 and 33, for which only 23 works. We also have two candidates for units digit 7: 27 and 37, for which only 37 works.

Thus, our answer is $2+3+5+7+23+37 = \boxed{77}$.

7. Let x and y be two numbers taken from the set $\{2^0, 2^1, 2^2, 2^3, \ldots, 2^{10}\}$ such that

$$xy - (x+y) = 104.$$

What is the value of $x+y$?

Answer: 24

Solution 1: Using Simon's Favorite Factoring Trick, we can write $xy-(x+y) = 104$ as $xy-(x+y)+1 = 105$, which is equivalent to $(x-1)(y-1) = 105$. Note $105 = 1 \times 105 = 3 \times 35 = 5 \times 21 = 7 \times 15$. After checking a couple powers of 2 for x and y, we see that the desired product is $(15)(7) = (2^4-1)(2^3-1) = 105$, so $x+y = 2^4+2^3 = \boxed{24}$.

Solution 2: Recognize that $128 = 2^7$ is the closest power of 2 to 104. The difference is $128-104 = 24$. Notice that 24 is indeed a sum of powers of 2 and that the exponents add up to 7: 2^3+2^4. Thus, $x+y = \boxed{24}$, as we hypothesized.

8. Thomas tosses a fair coin 3 times, and then Jefferson tosses the same coin 4 times. What is the probability that they flip exactly 3 heads in total?

Answer: $\mathbf{\frac{35}{128}}$

Solution 1: For Thomas, we have $2^3 = 8$ possible sequences of flips. We consider the cases of 0 heads, 1 head, 2 heads, and 3 heads. There is only 1 way to achieve all heads and 1 way to achieve no heads, and there are $\binom{3}{1} = \binom{3}{2} = 3$ ways to achieve 1 head or 2 heads.

For Jefferson, we have $2^4 = 16$ possible sequences of flips. We again consider the cases of 0 heads, 1 head, 2 heads, and 3 heads. There is only 1 way to achieve all heads or no heads, $\binom{4}{1} = \binom{4}{3} = 4$ ways to achieve 1 head or 3 heads, and $\binom{4}{2} = 6$ ways to achieve 2 heads. Let T represent the number of heads that Thomas flipped and let J represent the number of heads that Jefferson flipped. Then, we have that the probability that the number of total heads flipped is

$$\begin{aligned}
&\mathbb{P}[T=0, J=3] + \mathbb{P}[T=1, J=2] + \mathbb{P}[T=2, J=1] + \mathbb{P}[T=3, J=0] \\
&= \frac{1}{8} \times \frac{4}{16} + \frac{3}{8} \times \frac{6}{16} + \frac{3}{8} \times \frac{4}{16} + \frac{1}{8} \times \frac{1}{16} \\
&= \boxed{\frac{35}{128}}.
\end{aligned}$$

Solution 2: An equivalent framing of this problem is as follows: "what is the probability that upon flipping a coin 7 times, Thomas Jefferson flips 3 heads?" This yields $\binom{7}{3} = 35$ successful outcomes out of $2^7 = 128$ possible outcomes, for an answer of $\boxed{\frac{35}{128}}$.

9. Let O_n be the sum of the first n positive odd integers, and let E_n be the sum of the first n positive even integers. For example, $E_1 = 2$ and $O_2 = 1+3 = 4$. What is the value of the product

$$\left(\frac{O_1}{E_1}\right)\left(\frac{O_2}{E_2}\right)\cdots\left(\frac{O_{2024}}{E_{2024}}\right)?$$

Answer: $\mathbf{\frac{1}{2025}}$

Solution: Note that $O_n = n^2$ and $E_n = n(n+1)$. Simplifying, $\frac{n^2}{n(n+1)} = \frac{n}{n+1}$ for each term. Then we have $n = 1$ and

$$\left(\frac{O_1}{E_1}\right)\left(\frac{O_2}{E_2}\right)\cdots\left(\frac{O_{2024}}{E_{2024}}\right) =$$

$$\left(\frac{1}{1+1}\right)\left(\frac{2}{2+1}\right)\cdots\left(\frac{2024}{2024+1}\right) = \boxed{\frac{1}{2025}}.$$

10. In triangle $\triangle ABC$, points D and E are chosen on $\overline{AB}$ and $\overline{AC}$, respectively, such that $\overline{DE}$ is parallel to $\overline{BC}$. If $AE = BE$, $\angle BDE = 125°$, and $\angle ACB = 85°$, what is the value of $\angle BEC + \angle BAE$ in degrees?

Answer: **120°**

Solution: First, for ease of notation, define $\alpha = \angle BDE$ and $\beta = \angle AED$. Firstly, $\angle ADE = 180° - \alpha = 180° - 125° = 55°$, and from this, we can say $\angle DAE = 40°$. Also, since $AE = BE$, $\angle BAE = \angle ABE = 40°$. Define $\gamma = \angle BAE$.

Now, $\angle DEB$ can be calculated as $180° - \alpha - \gamma = 180° - 125° - 40° = 15°$, and with this, $\angle BEC$ can be computed to be $180° - \beta - \angle DEB = 180° - 85° - 15° = 80°$. Therefore, our final answer is $80° + 40° = \boxed{120°}$.

11. Find the sum of all two-digit positive integers, x, such that x is a factor of 2024 and the units digit of x^{2024} is the same as the units digit of x.

Answer: **57**

Solution: We first find the candidates for the units digit of possible values of x.

- For the digits $0, 1, 5$, and 6, raising them to any power will return $0, 1, 5$, and 6 respectively, so these four are all possibilities.
- If the factor has units digit 2, the units digit of the powers will repeat every four powers: $2, 4, 8, 6$. Thus, raising to the power of 2024 will not yield a units digit of 2 since 2024 is divisible by 4 (it wil yield a units digit of 6). The same is true for units digit 3, which has pattern $3, 9, 7, 1$; units digit 7, which has pattern $7, 9, 3, 1$; and units digit 8, which has pattern $8, 4, 2, 6$. Thus, all of these cases fail.
- If the factor has units digit 4, raising to odd powers will produce a units digit of 4 and raising to even powers will produce units digit 6. Units digit 9 has a similar pattern: odd powers produce a units digit of 9 while even powers produce a units digit of 1. Since 2024 is even, these cases also fail.

Thus, the units digit of x must be $0, 1, 5$, or 6. The prime factorization of $2024 = 2^3 \cdot 11 \cdot 23$. Therefore, we have only two possibilities: $x = 11$ or $x = 2 \cdot 23 = 46$.

Thus, our answer is $11 + 46 = \boxed{57}$.

12. What is the product of all real numbers, x, that satisfy

$$\frac{x^2 - 20x}{x^2 - 20x - 24} + \frac{x^2 - 20x + 24}{x^2 - 20x} = 2?$$

Answer: $\mathbf{-12}$

Solution: Let $a = x^2 - 20x$. Then,

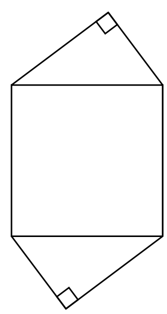
$$\frac{a}{a-24} + \frac{a+24}{a} = \frac{2a^2 - 576}{a^2 - 24a} = 2.$$

Solving for a, we find

$$\begin{aligned} 2a^2 - 576 &= 2a^2 - 48a \\ 48a &= 576 \\ a &= 12. \end{aligned}$$

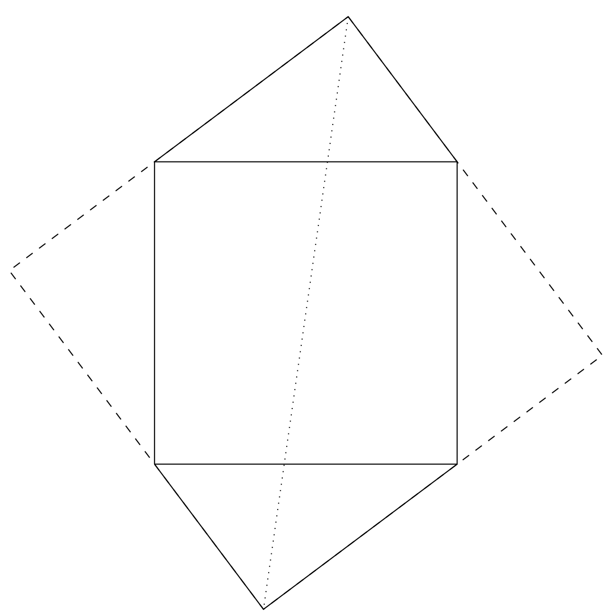
Thus, we would like the find the product of all real x such that $x^2 - 20x = 12$, or $x^2 - 20x - 12 = 0$. We can verify that this has two distinct real roots because $b^2 - 4ac = (-20)^2 - 4(-12) > 0$. By Vieta's formulas, the product of all possible real x is therefore $\boxed{-12}$.

13. Consider a hexagon with consecutive side lengths of 8, 6, 10, 8, 6, and 10, formed by attaching two right triangles with legs of length 6 and 8 to the top and bottom of a square, as shown in the diagram below. Compute the largest distance between two points in this hexagon.



Answer: $\mathbf{14\sqrt{2}}$

Solution:



The largest distance is between the top and bottom vertex of the top and bottom triangle, respectively. Drawing two other right triangles on the sides of the square gives a square of length $6+8=14$, meaning that the distance between these two points is $\sqrt{14^2+14^2}=\boxed{14\sqrt{2}}$.

14. Kaity creates a list of all four-digit integers, $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$, with distinct digits such that $3000 < \underline{A}\,\underline{B}\,\underline{C}\,\underline{D} < 6000$, $\underline{A}\,\underline{B}\,\underline{C}$ is divisible by 3, $\underline{B}\,\underline{C}\,\underline{D}$ is divisible by 4, $\underline{D}\,\underline{C}\,\underline{B}$ is divisible by 5, and $\underline{C}\,\underline{B}\,\underline{A}$ is divisible by 6. Find the median of Kaity's list.

Answer: 4534

Solution: Based on the second, third, and fourth divisibility conditions, we know that either $B=0$ or $B=5$ and that both A and D must be even. This means $A=4$ since our integer is restricted to be between 3000 and 6000. We proceed to work through the cases that $B=0$ and $B=5$, starting with the case of $B=0$. Given $A=4$, we are restricted to $C=2,5$, and 8 from the first divisibility condition. Through trial and error, and checking the second divisibility condition and the digit distinctiveness conditions, the only possibilities are $4028, 4052, 4056$. Now, let us consider the case where $B=5$. Then we are restricted to $C=0,3,6,9$. Again, through trial and error, this gives us seven cases: $4508, 4532, 4536, 4560, 4568, 4592, 4596$.

Therefore, we have the following ordered list:

$$4028, 4052, 4056, 4508, 4532, 4536, 4560, 4568, 4592, 4596.$$

The median is therefore the average of 4532 and 4536, which is $\boxed{4534}$.

15. A *permutation* of a set of n integers is any arrangement of its elements in a specific order. For example, permutations of the list $(1,2,3)$ include $(1,3,2)$ and $(3,2,1)$. Let a permutation $(a_1, a_2, a_3, \ldots, a_n)$ of the positive integers from 1 to n, inclusive, be a *katamari* if, for all $1 \leq i \leq n$, the inequality $a_i < 2 + a_1 + a_2 + a_3 + \cdots + a_{i-1}$ holds. For example, the permutation $(1,2,3,4,5,7,6)$ is a katamari but $(5,4,1,2,3)$ is not. Compute the number of permutations of $(1,2,3,\ldots,9)$ that are katamaris.

Answer: 1080

Solution: Since the right-hand sum starts at 2, we must have that $a_1=1$ and $a_2=2$. Thus, every katamari permutation begins with 1 and 2. After that, $2+a_1+a_2=5$ so we have the choice of 3 or 4 for a_3. If we choose $a_3=3$, then $4 \leq a_4 \leq 7$. After that, because $2+a_1+a_2+a_3+a_4 \geq 2+1+2+3+4=12$, there are no restrictions on any other a_i. Then meaning that there are $4 \cdot 5! = 480$ katamari permutations with $a_3=3$. Next, iff $a_3=4$, then $2+a_1+a_2+a_3=9$, so we must have $a_4=3$ or $5 \leq a_4 \leq 8$ and subsequently no restrictions on any other a_i, meaning that we have $5 \cdot 5! = 600$ choices. It follows that there are $480+600=\boxed{1080}$ permutations of $(1,2,3,\cdots,9)$ that are katamaris.

16. A positive integer is called *square-free* if it is not divisible by any perfect square greater than 1. Compute the sum of all positive integers, k, for which there exists a square-free positive integer, n, such that n^k has 64 divisors.

Answer: 74

Solution: Clearly 1^k does not have 64 divisors for any k, so consider $n = p_1p_2p_3\ldots p_s \neq 1$ for some distinct primes $p_1, p_2, \ldots, p_s$. We have $n^k = p_1^k p_2^k \ldots p_s^k$, which has $(k+1)(k+1)\ldots(k+1) = (k+1)^s$ divisors. Thus, in order for n^k to have 64 divisors, we must have $(k+1)^s = 64$. As $64 = 64^1 = 8^2 = 4^3 = 2^6$, we have there are four solutions for k, namely $64-1=63, 8-1=7, 4-1=3$, and $2-1=1$. Thus, the sum of all possible values for k is $63+7+3+1=\boxed{74}$.

17. Let a sequence of integers, $a_1, a_2, \ldots, a_n$, be *anti-consecutive* if $-2024 \leq a_1 \leq 2024$ and $a_i + a_{i+1} = i$ for all $1 \leq i \leq n-1$. Over all anti-consecutive sequences of length 501, compute the sum of all possible values of a_{501} that are divisible by 3.

Answer: 338175

Solution:

Note that we are solving the following system:

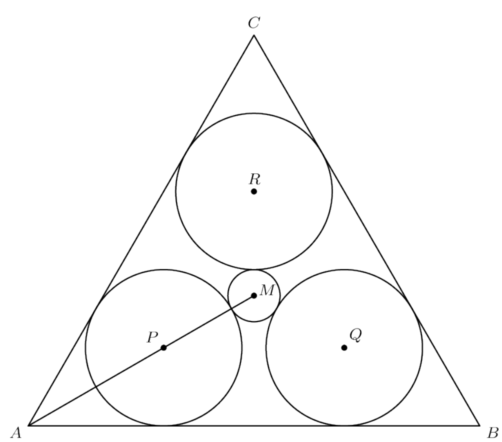
$$\begin{aligned} a_1 + a_2 &= 1, \\ a_2 + a_3 &= 2, \\ a_3 + a_4 &= 3, \\ &\vdots \\ a_{500} + a_{501} &= 500. \end{aligned}$$

We can set this up as a recursion $a_n + a_{n-1} = n - 1$, where we are solving for the nth integer. If we subtract the first equation from the second equation, we have $a_3 - a_1 = 1$. If we next subtract the third equation from the second equation, we get $a_5 - a_3 = 1$. If apply the same logic until a_{501}, then we have that $a_{501} = a_1 + 250$ with the added conditions that $-2024 \leq a_1 \leq 2024$ and a_{501} is divisible by 3. We find the lowest value for which this is satisfied is $a_1 = -2023$ and the highest possible value is $a_1 = 2024$. Plugging these in, we find the smallest possible value of a_{501} is -1773 and the largest possible value is 2274. Their sum is 501. Since we can increase the lower bound and decrease the upper bound by 3 each, the sum will remain constant. Letting $-2023 + 3(k-1) = 2024$, we find that there are $k = 1350$ possible values for a_1, which means there are $\frac{1350}{2} = 675$ pairs that sum to 501. Thus, our answer is $675 \cdot 501 = \boxed{338175}$.

18. Let $\triangle ABC$ be an equilateral triangle with side length 1. Let O_a, O_b, and O_c be three congruent circles inside $\triangle ABC$ such that O_a is tangent to $\overline{CA}$ and $\overline{AB}$, O_b is tangent to $\overline{AB}$ and $\overline{BC}$, O_c is tangent to $\overline{BC}$ and $\overline{CA}$, and none of the circles intersect each other. Let O_m be a circle inside $\triangle ABC$ that is externally tangent to O_a, O_b, and O_c. What is the minimum possible sum of the areas of O_a and O_m?

Answer: $\boldsymbol{\frac{\pi}{30}}$

Solution:



Let M be the center of $\triangle ABC$, let P be the center of O_a, let Q the center of O_b, and let R the center of O_c. Since the diagram is symmetrical, M is also the center of O_m. Furthermore, let r be the radius of the three congruent circles and s be the radius of the center circle. Since O_a is tangent to O_m, then $PM = r + s$, and since O_a is tangent to both $\overline{CA}$ and $\overline{AB}$, then $AP = 2r$. Therefore, $AM = AP + PM = 3r + s$. Since $\overline{AM}$ is also the circumradius of $\triangle ABC$, we have that $AM = \frac{1}{\sqrt{3}}$. Solving, we have $s = \frac{\sqrt{3}}{3} - 3r$.

We want to find the minimal value of $\pi r^2 + \pi s^2 = \pi(r^2 + s^2)$. Substituting our value of s in, this is equivalent to the minimal value of

$$\pi\left(r^2 + \left(\frac{\sqrt{3}}{3} - 3r\right)^2\right) = \pi\left(10r^2 - 2\sqrt{3}r + \frac{1}{3}\right).$$

Now, the minimum of this quadratic occurs at $-\frac{-2\sqrt{3}}{2(10)} = \frac{\sqrt{3}}{10}$, so the answer is

$$\pi\left(10\left(\frac{\sqrt{3}}{10}\right)^2 - 2\sqrt{3}\left(\frac{\sqrt{3}}{10}\right) + \frac{1}{3}\right) = \pi\left(\frac{1}{30}\right) = \boxed{\frac{\pi}{30}}.$$

19. Let S be the sum of the cubes of the divisors of 4200. Compute the last two digits of S.

Answer: 20

Solution: We factor 4200 as $2^3 \cdot 3^1 \cdot 5^2 \cdot 7^1$, so we can write any divisor d of 4200 in the form $d = 2^w \cdot 3^x \cdot 5^y \cdot 7^z$ where $w \in \{0,1,2,3\}$ and $x \in \{0,1\}$ and $y \in \{0,1,2\}$ and $z \in \{0,1\}$. So the desired sum is

$$S = \sum_{d|4200} d^3 = \sum_{w=0}^{3}\sum_{x=0}^{1}\sum_{y=0}^{2}\sum_{z=0}^{1}(2^w \cdot 3^x \cdot 5^y \cdot 7^z)^3 = \sum_{w=0}^{3}\sum_{x=0}^{1}\sum_{y=0}^{2}\sum_{z=0}^{1} 2^{3w} \cdot 3^{3x} \cdot 5^{3y} \cdot 7^{3z},$$

which can be factored as

$$S = \left(\sum_{w=0}^{3} 2^{3w}\right)\left(\sum_{x=0}^{1} 3^{3x}\right)\left(\sum_{y=0}^{2} 5^{3y}\right)\left(\sum_{z=0}^{1} 7^{3z}\right),$$

which is

$$S = (1 + 8 + 64 + 512)(1 + 27)(1 + 125 + 15625)(1 + 343).$$

Therefore, $S = (585)(28)(15751)(344)$. The last two digits of $585 \cdot 28$ are the last two digits of $85 \cdot 8 + 85 \cdot 20 \equiv 80 + 00 \equiv 80 \pmod{100}$. Similarly, the last two digits of $15751 \cdot 344$ are the last two digits of $51 \cdot 4 + 1 \cdot 40 \equiv 04 + 40 \equiv 44 \pmod{100}$.

Finally, our answer is the last two digits of $80 \cdot 44$. Evaluating modulo 100, we have $80 \cdot 44 \equiv 20 + 00 \equiv \boxed{20} \pmod{100}$.

Note: we actually don't need to calculate $5^6 = 15625$. All we need to know is that the last two digits of 5^6 is 25 (just like 5^i for all positive integers $i > 1$).

20. Jonathan has 46 indistinguishable blue balls, 3 indistinguishable red balls, and a green ball in a bin. He continuously draws balls from the bin without replacement until he draws the green ball. For instance, Jonathan might draw a red ball, followed by two blue balls, another red ball,

and then the green ball, completing the process. Compute the number of possible sequences of draws that are possible under these conditions.

Answer: 249899

Solution: We perform casework on the number of red balls that were removed. In particular, if we remove i red balls, there are $\binom{k+i}{i}$ different sequences of removed balls that also remove k blue balls before removing the green ball. Therefore, the total number of ways is $\binom{i}{i} + \binom{i+1}{i} + \cdots + \binom{i+46}{i} = \binom{i+47}{i+1}$ by the Hockey Stick identity. Thus, the total number of sequences is $\binom{47}{1} + \binom{48}{2} + \binom{49}{3} + \binom{50}{4} = \binom{51}{4} - \binom{46}{0} = \frac{51 \cdot 50 \cdot 49 \cdot 48}{24} - 1 = (51 \cdot 49) \cdot (50 \cdot 2) - 1 = (50^2 - 1) \cdot 100 - 1 = 249900 - 1 = \boxed{249899}$.