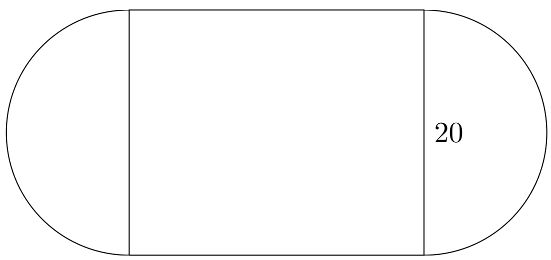
1. Find the positive integer x such that $x^x - \frac{12}{x^x} = 1$.

   Answer: 2

   Solution 1: Let $y = x^x$. In this case, $y - \frac{12}{y} = 1$. Multiplying both sides by y and rearranging yields $y^2 - y - 12 = 0$, or $(y+3)(y-4) = 0$, so $y = -3$ or $y = 4$. Since x^x is positive, this implies that $y = 4$, so $x^x = 2^2 = 4$, and $x = \boxed{2}$.

   Solution 2: Since the problem specifies that x is a positive integer, we can use trial and error by testing positive integers starting from 1. Plugging in $x = 1$ gives $1^1 - \frac{12}{1^1} = -11 \neq 1$, so it's not 1. Plugging in $x = 2$ gives $2^2 - \frac{12}{2^2} = 4 - \frac{12}{4} = 4 - 3 = 1$. So, the answer must be $x = \boxed{2}$.

2. A track is constructed by attaching two semicircular arcs with diameter 20 meters to a rectangle with width 20 meters and length 24 meters, as shown below. Preston runs a lap around the entire track, while Ronan only runs a lap around the rectangle. What is the absolute difference between the distance Preston runs and the distance Ronan runs, in meters?



   Answer: $20\pi - 40$

   Solution: Preston and Ronan both ran across the length of the rectangle, so the difference between distances they ran is the difference between the lengths of the circular parts of the track and the widths of the rectangle. The exteriors of two semicircles Preston ran both have length $\frac{1}{2}(20\pi)$, so he ran 20π meters that Ronan did not run. The widths Ronan ran are both 20 meters, so he ran 40 meters that Preston did not. Since $20\pi > 40$, Preston ran $\boxed{20\pi - 40}$ more meters than Ronan.

3. What is the sum of the odd factors of 2024?

   Answer: 288

   Solution: The prime factorization of 2024 is $2 \times 2 \times 2 \times 11 \times 23$. So, the odd factors are $1, 11, 23, 11 \times 23 = 253$, which sums to $\boxed{288}$.

4. Oski the bear wears a sock on each of his four distinguishable paws, where each paw has either a white sock or a black sock. If his four socks are not all the same color, find the number of different sock combinations Oski can wear.

   Answer: 14

   Solution: For each paw, there are 2 possible colors for the sock. Therefore, there are $2^4 = 16$ possible sock combinations. However, there are 2 combinations that are not allowed: the all-black and all-white sock combinations, so we ultimately have $16 - 2 = \boxed{14}$ combinations.

5. Eight friends are meeting up to share books with each other. Each friend brings some number of books, and the median number of books that the friends brought is 11. If six of the friends

bring $2, 10, 12, 16, 20,$ and 24 books, respectively, what is the maximum number of books the other two friends could have brought in total?

Answer: 20

Solution: If the number of books brought by all eight friends is put in order, the median, 11, must be between the fourth and fifth numbers. Since there are already 4 friemds among the first six that brought more than 11 books, the fourth largest number must be at least 12. However, note also that the fourth smallest number must be at least 10 since it is the second smallest current number, so it must be the case that the two middle numbers are exactly 10 and 12 for the median to be 11. Thus, neither of the last two friends could have brought more than 10 books. If both brought exactly 10 books, the number of books brought by each friend is $2, 10, 10, 10, 12, 16, 20, 24$, so the median is 11 as desired. Thus, $10 + 10 = \boxed{20}$ is the maximum number of books that could have been brought by the last two friends.

6. Austin writes the integers from 1 to 1000, inclusive, on a blackboard. He then erases every integer that is divisible by 5. How many instances of the digit 5 are left on the blackboard?

   Answer: 160

   Solution: We split the work into counting the number of occurrences place by place. Since every number with a 5 in the ones place is divisible by 5, there are zero occurrences of 5 in the ones place. For numbers where there is a 5 in the tens place, there are 10 digits that can be in the hundreds place and 8 digits in the ones place (note that 0 and 5 are discluded because they would make the number divisible by five). This gives us $10 \times 8 = 80$ occurrences of 5 in the tens places of all numbers. We can apply similar logic can be used for the occurrences of 5 in the hundreds place to figure out that there are 80 instances of 5 in the hundreds place. This gives us a total of $0 + 80 + 80 = \boxed{160}$ appearances of the digit 5 on the board.

7. Kiran has two buckets, Bucket A and Bucket B, each with lots of water, and a hose that can add water at a fixed rate to the buckets. Kiran punctures a hole into Bucket A so it leaks out water at a constant rate into Bucket B. If Kiran uses the hose to add water to Bucket A, Bucket A will first be holding 1 more gallon of water 48 seconds later. If he instead uses the hose to directly add water to Bucket B, Bucket B will first be holding 1 more gallon of water 16 seconds later. How many seconds does it take for a gallon of water to leak from Bucket A into Bucket B?

   Answer: 48

   Solution: Let d denote a gallon of watter, s denote the rate that the hose adds water to the buckets and w be the amount of water that leaks out of Bucket A, both in gallons per second. We can write two equations to represent that $\frac{\text{distance}}{\text{speed}} = \text{time}$, namely $\frac{d}{s-w} = 48$ and $\frac{d}{s+w} = 16$. After cross-multiplying, we get $16s + 16w = d$ and $48s - 48w = d$. From these equations, we can eliminate the s variable to get $w = \frac{d}{48}$, or $\frac{d}{w} = \boxed{48}$ seconds. Alternatively, this could be recognized as a harmonic mean problem, solving with $\frac{d}{w} = \frac{2 \cdot 48 \cdot 16}{48+16} = 24$, and substituting to get $\boxed{48}$ seconds.

8. Mary has a box that is 7 inches long and wide and 4 inches tall. What is the maximum number of 2 inch by 2 inch by 1 inch blocks she can fit inside the box without it overflowing?

   Answer: 48

   Solution: First, note that if any block is placed with sides not parallel to the sides of the box, it is impossible to tile the whole box with blocks as there will have to be polyhedra without parallel faces on the edges of the box, which can't have the same shape as the blocks we want to fit in the box.

If all blocks have sides parallel to the sides of the box, then we claim we will always miss out on at least 4 cubic inches of volume, meaning that our answer is at most 48 blocks. If we take a specific block and project it to one of the 2D faces of the box, it will always have even area: it either has a 2x2 profile or a 2x1 profile. In order to tile a $7 \times 7 \times 1$ region, the blocks must all be on the same layer, but projecting a valid tiling onto the 7×7 square would have the odd area of 49. This is impossible because the sum of all of the even areas would be even, meaning the maximal possible area is 48, so we are limited to 48 units of volume per $7 \times 7 \times 1$ layer.

Now, we show that you can actually fit 48 blocks in the box: In order to do so, place 9 blocks in a $6 \times 6 \times 1$ grid, and stack 4 such grids on top of each other to fit in a $6 \times 6 \times 4$ volume. Then, we have two $6 \times 1 \times 4$ slices remaining which can each fit 6 more slices, making a total of $36 + 2 \cdot 6 = \boxed{48}$ blocks fitting in the box, which is our answer.

9. A deck of eight distinct cards is shuffled in a particular way. In one round of shuffling, the following is done in order:

- The second card from the top is moved to the top.
- The third card from the top is moved to the top.
- The fifth card from the top is moved to the top.
- The seventh card from the top is moved to the top.

What is the least number of rounds of shuffling that would return the deck to its original order?

Answer: 6

Solution: After one round of shuffling, the cards are ordered as: $7, 5, 3, 2, 1, 4, 6, 8$. The third and eight cards stay in their original positions, while $1 \to 5 \to 2 \to 4 \to 6 \to 7 \to 1$ move about the deck in a cycle. Thus, it takes $\boxed{6}$ rounds of shuffling to complete the cycle and return the deck to its original order.

10. Recall that the arithmetic mean of n numbers $a_1, \ldots, a_n$ is $\frac{a_1+\cdots+a_n}{n}$ and the geometric mean of n numbers $a_1, \ldots, a_n$ is $\sqrt[n]{a_1 \ldots a_n}$. A five-element arithmetic sequence a_1, a_2, a_3, a_4, a_5 has a common difference of y and an arithmetic mean of x. A five-element geometric sequence b_1, b_2, b_3, b_4, b_5 has a common ratio of y and a geometric mean of x. If $a_1 = b_1 = 2024$, compute the sum of all possible values of y.

Answer: $\mathbf{\frac{1}{1012}}$

Solution: The arithmetic mean of the first sequence is:

$$\frac{a_1 + a_2 + a_3 + a_4 + a_5}{n} = a_1 + 2y,$$

and the geometric mean of the second sequence is:

$$\sqrt[5]{b_1 b_2 b_3 b_4 b_5} = y^2 b_1.$$

Since the two expressions above are both equal to the same value x, they must be equal. Substituting in $a_1 = b_1 = 2024$ yields:

$$2024y^2 - 2y - 2024 = 0$$

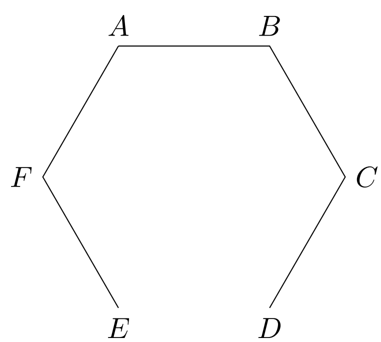
By Vieta's formula, the sum of all solutions y to the quadratic above is $\boxed{\frac{1}{1012}}$.

11. Jessica has an analog clock on her wall. At a certain time, she measures the smaller angle between the hour and minute hand and finds that it is 138°. She also has a digital clock on her nightstand showing the same time, and the product of the number of minutes and hours displayed is 72. What is the least number of minutes that could have passed since 12:00 A.M. of the same day?

    Answer: 156

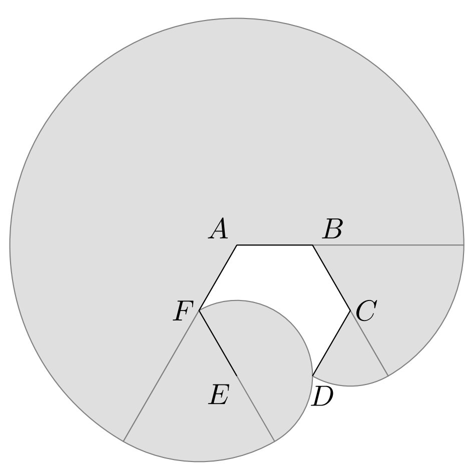
    Solution: The degree measure of the smaller angle can be calculated as $\min(30h - \frac{11m}{2}, \frac{11m}{2} - 30h)$, where $0 \le m < 60$ is the minutes in the hour and h is the number of hours. From the digital clock, we have $m \times h = 72$, or $m = \frac{72}{h}$. Rearranging and plugging in, we have either $30h - \frac{11\cdot 72}{2h} = 138$ or $\frac{11\cdot 72}{2h} - 30h = 138$. If we multiply both sides of both equations by h and solve as quadratics, we get the values h is one of ± 2 or $\pm\frac{33}{5}$. The digital clock can only display positive integers, so the only valid value of h is 2. Substituting to find m, we have $m = 36$, so the current time is 2:36. Clearly it is optimal to have this be 2:36 A.M., which is $2 \cdot 60 + 36 = \boxed{156}$ minutes after 12:00 A.M.

12. A dog is on a leash that is attached to the exterior of a fenced region in the shape of a regular hexagon, $ABCDEF$, with side length 1 foot. The dog's leash is 3 feet long and is attached to vertex A. If side $\overline{DE}$ is removed, as shown below, what is the area of the region the dog can reach while still attached to the leash?



    Answer: 8π

    Solution:



The region that the dog can reach while attached to the leash is shown above and has been split into five sectors. The largest sector with radius 3 has area $\frac{2}{3}\pi(3)^2 = 6\pi$ and the two sectors with radius 2 both have area $\frac{1}{6}\pi(2)^2 = \frac{2}{3}\pi$. The half disk (a semicircle with its interior included) has area $\frac{1}{2}\pi$, and the other sector with radius 1 has area $\frac{1}{6}\pi$. Thus, the total area the dog can reach is $6\pi + 2 \cdot \frac{2}{3}\pi + \frac{1}{2}\pi + \frac{1}{6}\pi = \boxed{8\pi}$.

13. Compute the sum of all values of c for which there exists a function f such that for all real numbers a and b, the following hold:

$$f(a) + f(a^2) = c,$$
$$f(0) + f(1-b) + f(1-2b+b^2) = c + 20.$$

Answer: 40

Solution: Given that the equations have to hold for all values of a and b, we can plug in various values. First, plugging $a = 0$ into the first equation, we know that $f(0) + f(0) = 2f(0) = c$, so $f(0) = \frac{c}{2}$. Then, plugging in $b = 1$ into the second equation, we have that $f(0) + f(0) + f(0) = 3f(0) = c + 20$. Plugging in $f(0) = \frac{c}{2}$ we have that $\frac{3c}{2} = c + 20$, so $c = 40$ is the only possible value of c.

Of course, $c = 40$ is possible if we simply let $f(x) = 20$ for all x. Thus, the answer is simply $\boxed{40}$.

14. Five people, Alice, Benji, Carl, Danielle, and Erica, are passing around a bag with 6 candies in it. Alice initially holds the bag and takes a candy from the bag. Then, she passes the bag and remaining candies to one of the other four people uniformly at random. That person takes a candy from the bag and then passes the bag and remaining candies to one of the other four people uniformly at random. This continues until there are no more candies in the bag. What is the probability that every person takes a candy from the bag?

Answer: $\mathbf{\frac{15}{64}}$

Solution: The number of sequences of five passes of the bag is 4^5, since each person has four options for where to pass the bag. Consider the number of these sequences where at least one person never touches the bag. Alice will always pass the bag since she starts with it, and someone will receive the bag next, so at most 3 people will not receive the bag. Below, we consider the cases where one, two, or three people don't receive the bag:

- First, we consider the case that one specific person doesn't receive the bag. Without loss of generality, suppose that Benji never receives the bag. In this case, whenever someone has the bag, they have 3 options for where to pass the bag, so there are 3^5 sequences of passes where Benji never receives the bag. This could happen for any of the 4 people who are not Alice.
- Next, we consider the case that two people don't receive the bag. Without loss of generality, suppose that Benji and Carl never receive the bag. In this case, each person has 2 options for where to pass the bag, so there are 2^5 sequences of passes where Benji and Carl never receive the bag. This could occur for any of the $\binom{4}{2} = 6$ pairs of people that do not include Alice.
- Lastly, we consider the case that three people never receive the bag. Without loss of generality, suppose that Benji, Carl, and Danielle never receive the bag. In this case, Alice and Erica must pass to each other, so there is only 1 sequence of passes with achieves this. This can happen to any of the 4 groups of 3 people that don't include Alice.

By the Principle of Inclusion and Exclusion, there are

$$4 \cdot 3^5 - 6 \cdot 2^5 + 4 \cdot 1 = 784$$

ways for at least one of the people to never have the bag, so there are $1024 - 784 = 240$ sequences of passes where all five people have held the bag and taken a chocolate. Since each sequence of passes is equally likely, the probability that every person takes a chocolate at least once is $\frac{240}{1024} = \boxed{\frac{15}{64}}$.

15. Let S be the sum of all positive integers $n \leq 91$ such that $n^2 + 2n + 18$ is divisible by 91. Compute the remainder when S is divided by 91.

Answer: 87

Solution: We consider this equation both modulo 7 and modulo 13. First, we have

$$n^2 + 2n + 18 \equiv n^2 + 9n + 18 = (n+3)(n+6) \pmod 7.$$

Therefore the solutions must be 1 or 4 $\pmod 7$. Then we have

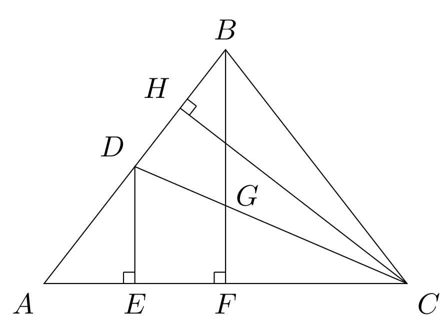
$$n^2 + 2n + 18 \equiv n^2 - 11n + 18 = (n-2)(n-9) \pmod{13}$$

So we get that our solutions must be 2 or 9 $\pmod{13}$. Therefore, we have four solutions with every possible combination of residues mod 7 and 13. We can add two copies of each residue to get the residue of S modulo 7 and 13. We obtain that $S \equiv 3 \pmod 7$ and $S \equiv 9 \pmod{13}$. We see that these are both -4 in their respective residues, so $S \equiv -4 \equiv \boxed{87} \pmod{91}$.

16. Triangle $\triangle ABC$ has point D on side $\overline{AB}$ so that $AD = BD$ and $BC = CD$. Point E is placed on side $\overline{AC}$ so that $\angle ABC = 2\angle ADE$. If $AE = 4$ and $CE = 12$, find BC.

Answer: $\frac{16\sqrt{6}}{3}$

Solution:



Draw a line parallel to $\overline{DE}$ through B that intersects side $\overline{AC}$ at F and segment $\overline{CD}$ at G. Then, $\triangle AED$ is similar to $\triangle AFB$, with the side lengths of $\triangle AFB$ being double the lengths of $\triangle AED$ (since $AB = AD + BD = 2AD$). Therefore,

$$AF = 2AE = 8,$$

$$CF = AE + CE - AF = 4AE - 2AE = 8.$$

This makes $\overline{BF}$ the median of $\triangle ABC$ from $\angle ABC$. Also, since $\measuredangle ABF = \measuredangle ADE = \frac{1}{2}\measuredangle ABC$, we know that $\overline{BF}$ is also the angle bisector of $\angle ABC$. Thus, the angle bisector of $\angle ABC$ is also its median, which means $\triangle ABC$ is isosceles, with $AB = BC$.

Now, drop an altitude from C to $\overline{AB}$, and let it intersect $\overline{AB}$ at H. Since $\triangle BCD$ is isosceles with $BC = CD$, this means that $\overline{CH}$ is the median from $\angle C$, so $BH = DH$. Furthermore, since $\triangle AED \sim \triangle AFB$ with side lengths half as long, we know that $AD = BD$ and so $AB = AD + BH + DH = 4DH$. But, since $\triangle ABC$ is isosceles, we know that $AB = BC$, and therefore $DH = \frac{1}{4}BC$.

Finally, we can compute BC using two right triangles: $\triangle CHD$ with leg lengths CH and $DH = \frac{1}{4}BC$ and hypotenuse length $CD = BC$, and $\triangle CHA$ with leg lengths CH and $HA = DH + AD = \frac{3}{4}BC$ and hypotenuse length $AC = 4AE = 16$. By the Pythagorean Theorem, we get:

$$CH^2 + \frac{BC^2}{16} = BC^2,$$

$$CH^2 + \frac{9BC^2}{16} = 16^2.$$

Subtracting the first equation from the second gives $\frac{BC^2}{2} = 16^2 - BC^2$, which tells us that $\frac{3}{2}BC^2 = 16^2$, or $BC = 16\sqrt{\frac{2}{3}} = \boxed{\frac{16\sqrt{6}}{3}}$.

17. Tushar has three fair 20-sided dice, D_1, D_2, and D_3. Die D_1 has faces labeled with the integers from 1 to 20, inclusive, die D_2 has faces labeled with the even integers from 2 to 40, inclusive, and die D_3 has faces labeled with the multiples of 3 from 3 to 60, inclusive. Tushar selects one of these three dice from a bag uniformly at random and then rolls the selected die three times. If the sum of Tushar's three rolls is 30, what is the probability that he selected D_1?

Answer: $\mathbf{\frac{298}{425}}$

Solution: If the average of the 3 rolls is 10, then the sum of all 3 rolls is 30. D_1, D_2, D_3 all have the same number of faces, so each has the same number of possible sequences of three rolls, $20^3 = 8000$. Moreover, since each die is equally likely to be chosen initially, it is only necessary to compare the number of ways three rolls of each die could result in a sum of 30.

For D_1, each roll is at least 1. If we ignore the fact that any given roll is at most 20, we can treat the number of ways for three rolls to sum to 30 to be equivalent to the number of ways to distribute $30 - 3 = 27$ balls into 3 bins. This is equal to the number of permutations of 27 objects and 2 dividers, or $\binom{29}{2} = \frac{29 \cdot 28}{2} = 406$.

Now, we account for die values being at most 20. Since this value includes situations where an individual die roll exceeds 20, those cases must be subtracted. The possibilities are the cases when 20 to 27 balls land in the same bin. Notably, this can only be true for one of the bins, since there are only 27 balls. Once a bin has some $x \geq 20$ balls in it, there are $27 - x + 1$ ways to distribute the remaining $27 - x$ balls among the remaining two bins. Thus, the total number of cases to subtract is $3 \cdot (1 + 2 + \ldots + 8) = 108$, so the total number of sequences of rolls of D_1 that result in an average of 10 is $406 - 108 = 298$.

Next, we look at the other two dice. Since each face of D_2 is divisible by 2, the number of ways for three rolls of D_2 to have a sum of 30 is equal to the number of ways for three rolls of D_1 to have a sum of 15. Using the same balls and bins counting as before, the total number of sequences of rolls is $\binom{14}{2} = 91$. Similarly, the number of ways for three rolls of D_3 to have a sum of 30 is equal to the number of ways for three rolls of D_1 to have a sum of 10. In this case, there are $\binom{9}{2} = 36$ ways for this to occur.

In total, there are $298 + 91 + 36 = 425$ ways for the average of the three die rolls to be 10, and 298 of them are from rolling D_1. Thus, because we pick the dice with equal chance, the probability that Tushar selected D_1 is $\boxed{\frac{298}{425}}$.

18. Let p, q, and r be the three roots of the cubic polynomial $x^3 - 5x + 1$. Compute

$$\frac{1}{p^3+5} + \frac{1}{q^3+5} + \frac{1}{r^3+5}.$$

Answer: $\mathbf{\frac{7}{51}}$

Solution: First, because p, q, and r are roots of $x^3 - 5x + 1$, then $p^3 = 5p - 1$ and likewise for q and r. Thus, we can plug in these values and then manipulate the expression:

$$\begin{aligned}\frac{1}{p^3+5} + \frac{1}{q^3+5} + \frac{1}{r^3+5} &= \frac{1}{5p+4} + \frac{1}{5q+4} + \frac{1}{5r+4} \\ &= \frac{(5q+4)(5r+4) + (5p+4)(5r+4) + (5p+4)(5q+4)}{(5p+4)(5q+4)(5r+4)} \\ &= \frac{25(pq+pr+qr) + 40(p+q+r) + 48}{125pqr + 100(pq+pr+qr) + 80(p+q+r) + 64}.\end{aligned}$$

By Vieta's formulas, $p + q + r = 0$, $pq + pr + qr = -5$, and $pqr = -1$. Plugging this in gives us

$$\frac{25(pq+pr+qr) + 40(p+q+r) + 48}{125pqr + 100(pq+pr+qr) + 80(p+q+r) + 64} = \frac{25(-5) + 40(0) + 48}{125(-1) + 100(-5) + 80(0) + 64} = \boxed{\frac{7}{51}}.$$

19. A right cylinder is placed in a cube with side length 1 such that it is tangent to all 6 faces of the cube and its altitude is parallel to a main diagonal of the cube. What is the height of the cylinder, given that it is twice the cylinder's diameter?

Answer: $\mathbf{2\sqrt{3} - \sqrt{6}}$

Solution 1: Let r denote the radius of the cylinder, in which case the height of the cylinder is $4r$. Consider a plane that passes one of the bases of the cylinder, and let O denote the center of this base. Let point A be the vertex of the cube closest to the this base. This plane intersects three edges of the cube at points equidistant from A by symmetry. Let B, C, and D denote these points.

Tetrahedron $ABCD$ has the following characteristics:

- Triangle $\triangle BCD$ is equilateral with incircle O.
- $\angle BAC = \angle CAD = \angle BAD = 90°$.

Let E denote the midpoint of $\overline{BC}$. Since triangle $\triangle BOE$ is a 30-60-90 triangle, $BE = \sqrt{3}r$ and $BC = 2\sqrt{3}r$. Since $AB = AC$, and $\angle BAC = 90°$, and $AE = \frac{\sqrt{3}}{r}$, so $AO = \sqrt{2}r$.

Now, the main diagonal of the cube has length $\sqrt{3}$, so $4r + 2AO = 4r + 2\sqrt{2}r = \sqrt{3}$. This means that $r = \frac{\sqrt{3}}{4+2\sqrt{2}} = \frac{4\sqrt{3}-2\sqrt{6}}{8}$. Thus, the height is of the cylinder is $4r = 2\sqrt{3} - \sqrt{6}$.

Solution 2: The first step is to figure out where the cylinder is tangent to the cube. Consider one corner for now. By symmetry, the tangency points are either on the 3 edges meeting at the corner, or on the 3 faces meeting at the corner. It can't be at the edges, as drawing the circle through the equilateral triangle formed by the tangency points shows that it juts out of the cube. Hence, it must be tangent to the faces. By symmetry, these tangency points must lie on the face diagonals on each face from the corner to the opposing corner. But note that one can form a regular tetrahedron with the 6 full face diagonals. Since the 3 tangency points lie on 3 of the face diagonals, there exists a dilation centered at the corner that takes our smaller equilateral triangle, to the larger one that is part of a large, cube-spanning regular tetrahedron. Hence, we have a small regular tetrahedron resting on top of the cylinder. Using this information, we aim to compute the height of the cylinder.

Let x and $2x$ be the diameter and height of the cylinder, respectively, so that we seek to find $2x$. By symmetry, we can consider half of the space diagonal and its length, which is $\frac{\sqrt{3}}{2}$. We observe that the length of half of the space diagonal consists of the height x of half of the whole cylinder plus the height h of the small regular tetrahedron, meaning that $h + x = \frac{\sqrt{3}}{2}$. Additionally, the circle that forms one face of the cylinder has radius $\frac{x}{2}$ and the equilateral triangle inscribed in the circle has side length $\frac{x\sqrt{3}}{2}$. Using this, the height h of a regular tetrahedron can be computed by considering the altitude, which by symmetry must have base at the center of the circle. This gives $h = \sqrt{\left(\frac{x\sqrt{3}}{2}\right)^2 - \left(\frac{x}{2}\right)^2} = \frac{x}{2}\sqrt{3-1} = \frac{x\sqrt{2}}{2}$, and so

$$\frac{\sqrt{3}}{2} = h + x = x + \frac{x\sqrt{2}}{2} = x\left(1 + \frac{\sqrt{2}}{2}\right),$$

which means that

$$2x = \frac{\frac{2\sqrt{3}}{2}}{1 + \frac{\sqrt{2}}{2}} = \frac{2\sqrt{3}}{2+\sqrt{2}} = \boxed{2\sqrt{3} - \sqrt{6}}.$$

20. Shreyas is playing a game. He begins with 2 marbles in his collection. Then, on turn n of the game, he flips a fair coin and adds 2^n marbles to his collection if it lands heads. The game ends when the number of marbles in his collection is divisible by either 3 or 5. What is the probability that the number of marbles in his collection is divisible by 5 at the end of the game?

Answer: $\mathbf{\frac{2}{7}}$

Solution: Let x denote the number of marbles Shreyas has in his collection at some point during the process. In this case, the probability that the final number of marbles is divisible by 5 is the probability that $x \equiv 0 \pmod 5$ occurs no later than when $x \equiv 0 \pmod 3$ occurs. By the Chinese Remainder theorem, $x \pmod 3$ and $x \pmod 5$ are uniquely defined by $x \pmod{15}$. If Shreyas adds some 2^n coins to his collection, the number of coins Shreyas adds to his collection is equivalent to one of 2, 4, 8, or $16 \equiv 1 \pmod{15}$, meaning that we only care about the value of n modulo 4.The table below shows their corresponding values modulo 3 and modulo 5.

mod 15	2	4	8	1
mod 3	2	1	2	1
mod 5	2	4	3	1

Initially, Shreyas has $2 \pmod{15}$ coins in his collection. Consider the next flip, n, that lands heads. If $n \equiv 2 \pmod 4$ or $n \equiv 0 \pmod 4$, the number of coins in Shreyas's collection is divisible by 3, so the process ends and the number of coins is not divisible by 5. If $n \equiv 3 \pmod 4$, the process ends and the number of coins in Shreyas's collection is divisible by 5.

The only other case to consider is where $n \equiv 1 \pmod 4$, in which case the number of coins in Shreyas's collection is equivalent to $4 \pmod{15}$, which is not divisible by 3 or 5. Given that $n \equiv 1 \pmod 4$, let m denote the number of flips after flip n where Shreyas adds coins to his collection for the second time. Notably, $n + m \equiv m + 1 \pmod 4$. Thus, if $m \equiv 2 \pmod 4$ or $m \equiv 0 \pmod 4$, the process ends and the number of coins is not divisible by 5, and if $m \equiv 3 \pmod 4$, the process ends and the number of coins is divisible by 5. Similarly, if $m \equiv 1 \pmod 4$, the number of coins in Shreyas's collection is equivalent to $8 \pmod{15}$, which is divisible by neither 3 or 5. This pattern, where the number of flips between two consecutive heads is equivalent to $1 \pmod 4$, continues.

So, to summarize, if $n \equiv 1 \pmod 4$, the situation that we are now in is effectively identical to our original situation.

The probability that $n \equiv 1 \pmod 4$ is $\frac{1}{2} + \left(\frac{1}{16}\right)^1\left(\frac{1}{2}\right) + \left(\frac{1}{16}\right)^2\left(\frac{1}{2}\right) + \ldots = \frac{8}{15}$. The probability that $n \equiv 3 \pmod 4$ is $\frac{1}{8} + \left(\frac{1}{16}\right)^1\left(\frac{1}{8}\right) + \left(\frac{1}{16}\right)^2\left(\frac{1}{8}\right) + \ldots = \frac{2}{15}$. Let p equal the probability that the number of coins after the process is divisible by 5. In this case, $p = \frac{8}{15}p + \frac{2}{15}$, in which case $p = \boxed{\frac{2}{7}}$.