GEOMETRY ANSWER KEY

1. $\dfrac{\pi\sqrt{3}}{6}$
2. $\dfrac{3\sqrt{3}}{2}$
3. $\sqrt{2}$
4. $7 + 6\sqrt{2} + 4\sqrt{3}$
5. $\dfrac{5}{12}$
6. 1
7. $\dfrac{3(1+\sqrt{5})}{2}$
8. $\dfrac{9}{2}\pi$
9. $36\sqrt{3}$
10. $\dfrac{825}{128}$

[**P1.**] Let ABC be a triangle. Let r denote the inradius of $\triangle ABC$. Let r_a denote the A-exradius of $\triangle ABC$. Note that the A-excircle of $\triangle ABC$ is the circle that is tangent to segment BC, the extension of ray AB beyond B and the extension of AC beyond C. The A-exradius is the radius of the A-excircle. Define r_b and r_c analogously. Prove that

$$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}.$$

SOLUTION: Lets use the notation $[XYZ]$ is the area of triangle XYZ, s is the semiperimeter, $a = BC$, $b = CA$, and $c = AB$. Note the formulas: $A = rs = r_a(s - a)$. Let us prove that $A = r_a(s-a)$. Let I_A denote the center of the A excircle. Notice that $[ABC] = [I_ACA] + [I_ABA] - [I_ABC] = \frac{1}{2}(r_a \cdot b + r_a \cdot c - r_a \cdot a) = r_a(s - a)$. Using these formula, we get that the result is equivalent to $s = (s - a) + (s - b) + (s - c)$, which is true.

- 4 points for proving that $\frac{r_a}{r} = \frac{s}{s-a}$ (or equivalent). There are at least two possible approaches:
  1. Proving the formulas $A = rs$, $A = r_a(s-a)$. -2 points if $A = r_a(s-a)$ is stated without proof.
  2. Say the incircle is tangent to AB at X and the A-excircle is tangent to line AB at X'. Then $AX/AX' = r/r_a$. 2 points for calculating AX correctly using equal tangents. 2 points for calculating AX' correctly using equal tangents.
- 2 points for complete solution.

[**P2.**] Let ABC be a fixed scalene triangle. Suppose that X, Y are variable points on segments AB, AC, respectively such that $BX = CY$. Prove that the circumcircle of $\triangle AXY$ passes through a fixed point other than A.
SOLUTION: Without loss of generality, suppose that $AB < AC$. Let the perpendicular bisector of segment BC intersect arc BAC at P. As $AB < AC$, X lies on minor arc AC. Choose some value of $BX = CY$. Observe that $PB = PC$ and $BX = CY$. Further, $\angle PBX = \angle PBA = \angle PCA = \angle PCY$. Thus $\triangle XBP = \triangle YCP$ by SAS similarity. It follows that $\angle XPB = \angle YPC$. Now we prove that $APYX$ is cyclic. Indeed, $\angle XAY = \angle BAC = \angle BPC = \angle BPY + \angle YPC = \angle PBY + \angle XPB = \angle XPY$. Thus $APYX$ is cyclic. Evidently, P is a fixed point, and the circumcircle of $\triangle AXY$ passes through P, so we are done.

- 2 points for correctly recognizing that the other fixed point lies on the circumcircle of $\triangle ABC$ and the perpendicular bisector of segment BC.
- 1 point for proving that $\triangle XBP = \triangle YCP$.
- 1 point for proving that P lies on the circumcircle of $\triangle AXY$.
- 2 points for a fully correct solution, which contains the following elements: clearly explaining why P is a fixed point, being clear about any possible configuration issues (eg. stating WLOG $AB < AC$, using directed angles)

Comments: Other solutions may be possible. Indeed, one can try to define P as the intersection of two circles and show that P lies on the perpendicular bisector of segment BC. -1 point if one has a fully correct solution along these lines except the solver does not explain why P lies on arc BAC as opposed to the other arc with endpoints at B, C. Perhaps other solutions are possible along the lines of showing that the line connecting center of $C(AXY)$ and the circumcenter of $C(ABC)$ is a fixed line.