Introduction

In this round, we will be explicitly focused on analyzing a function known as the **arithmetic derivative**. While the name is rooted in calculus, this function is purely a number theoretic phenomena. Nevertheless, it shares some interesting mathematical characteristics with its calculus counterpart. In this round, we will be looking at the function definition, examine bounds on the function, and use the function definition to solve arithmetic differential equations. The arithmetic derivative has also been shown to give insight into abstract number theory problems.

Definition. The **Arithmetic Derivative**, $D : \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined as follows:

- $D(0) = 0$
- $D(p) = 1$ if p is prime
- *Product Rule:* If $n = ab$ for $a, b \in \mathbb{N}$, then $D(n) = aD(b) + bD(a)$.

Example. For example, if we wanted to calculate the arithmetic derivative of 10, we would see that $10 = 5 \cdot 2$. So $D(10) = 5D(2) + 2D(5)$ which is $5 \cdot 1 + 2 \cdot 1 = 7$.

Since this function deals with primes, let's start our analysis of this function by looking at prime powers.

A Note on Proofs

Many of these proofs require the use of induction, a proof technique that is very common in number theory. To prove a statement P that depends on a variable n in the natural numbers (i.e 1, 2, 3 ...) is true using induction, first prove that P is true for 1. Then show that if P is true for a natural number k, then P is true for the natural number $k - 1$. An example is illustrated below.

Example. Prove that $\sum_{i=1}^{n} 2 \cdot i - 1 = n^2$. In other words show that the sum of the first n odd numbers is n^2.

Proof. We proceed using induction. Our statement $P(n)$ is "the sum of the first n odd numbers is n^2" First we will prove the base case or $P(1)$. In effect this means we need to show that the sum of the first odd number is 1^2, which is clearly true since $1 = 1$.

Now, we will show that if the statement "the sum of the first n odd numbers is n^2" is true for $n = k$, then it is true for $n = k + 1$. So we assume that the sum of the first k odd numbers is k^2. So we have $\sum_{i=1}^{k} 2 \cdot i - 1 = k^2$. Adding the $k + 1$st odd number or $2(k + 1) - 1 = 2k + 1$ to the sum, we see that $\sum_{i=0}^{k+1} 2 \cdot i - 1 = k^2 + 2k + 1 = (k + 1)^2$. Note that this is exactly in the same thing as saying "the sum of the first $k + 1$ odd numbers is $(k + 1)^2$, so we have shown that $P(k + 1)$ is true given $P(k)$ is true.

Therefore, since we have shown that $P(1)$ is true and that $P(k)$ implies $P(k+1)$, we have shown that the statement P is true on the natural numbers.

□

1. **[2 pts]** Prove that $\sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}$.

   Solution to Problem 1:

   We proceed with induction. Our base case is that the sum of $1^2 = \frac{1 \cdot 2 \cdot 3}{6} = 1$, so we are done. We will now proceed with our induction step. Assume that for $n = k$, $\sum_{i=1}^{k} i^2 =$

$\frac{k(2k+1)(k+1)}{6}$. Adding $(k+1)^2$ to both sides, we have $\sum_{i=1}^{k+1} i^2 = \frac{k(2k+1)(k+1)}{6} + (k+1)^2 = \frac{k(2k+1)(k+1)+6(k+1)^2}{6} = \frac{(k(2k+1)+6(k+1))(k+1)}{6} = \frac{(2k^2+7k+6)(k+1)^2}{6} = \frac{(k+2)(2k+3)(k+1)^2}{6} = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$ which is exactly what we wanted to prove.

The Arithmetic Derivative on Prime Powers

2. **[1 pt each]** Find the arithmetic derivative of the following prime powers. You must show your work.

   (a) 5^3

   (b) 11^5

   (c) Find and prove a formula for the arithmetic derivative of p^4 for any prime p.

   Solution to Problem 2:

   (a) $D(5^3) = 5D(5^2) + 5^2D(5) = 5(5D(5) + 5D(5)) + 5^2D(5) = 5(10) + 25(1) = \boxed{75}$

   (b) We first calculate $D(11)$, $D(11^2)$ and $D(11^3)$. $D(11) = 1$. $D(11^2) = 11 \cdot D(11) + 11 \cdot D(11) = 22$. Finally, $D(11^3) = 11^2D(11) + 11D(11^2) = 121 + 242 = 363$. Now $D(11^5) = 11^3D(11^2) + 11^2D(11^3) = 1331 \cdot 22 + 121 \cdot 363 = \boxed{73205}$.

   (c) We will show that $D(p^4) = 4p^3$. $D(p^4) = p^2D(p^2) + p^2D(p^2) = 2p^2D(p^2)$. Now, $D(p^2) = pD(p) + pD(p) = 2pD(p)$. So, $D(p^4) = 2p^2 \cdot 2p \cdot D(p) = \boxed{4p^3}$.

In general, we can express the arithmetic derivative of prime powers using what is known as the *power rule.*

Theorem. *Suppose natural number $n = p^k$ for some prime p and nonnegative integer k. Then, $D(p^k) = kp^{k-1}$.*

3. Let's try to prove the above theorem true.

   (a) **[1 pt]** Show that $D(1) = 0$.

   (b) **[4 pts]** Prove the above theorem. *Hint:* Use induction.

   Solution to Problem 3:

   (a) We see that $D(p) = 1$. But we also know that $D(p) = 1 \cdot D(p) + p \cdot D(1)$ So, $D(p) = D(p) + p \cdot D(1)$ which implies that $D(1) = 0$.

   (b) We will proceed with induction on k. Our base case is $k = 0$, which occurs when we wish to calculate $D(p^k) = D(p^0) = D(1)$. The previous part already showed us that $D(1) = 0$, and $0 = 0 \cdot p^{-1}$ for any prime, so we are done.

   Now, we assume that for $k = j - 1$, that $D(p^{j-1}) = j - 1p^{j-2}$. We will show that given this assumption, the case $k = j$ holds. We know that $D(p^j) = p \cdot D(p^{j-1}) + p^{j-1} \cdot D(p) = p \cdot (j-1)p^{j-2} + p^{j-1}$. This is $((j-1)+1)p^{j-1} = jp^{j-1}$, which is what we wanted. Therefore, the theorem is true.

The Arithmetic Derivative On All Integers

4. Let us compute the arithmetic derivative on general numbers.

   (a) **[1 pt]** Compute $D(899)$

   (b) **[1 pt]** Compute $D(36)$

   (c) **[1 pt]** We saw that the power rule states that $D(p^k) = kp^{k-1}$ if p is a prime. Give a counter-example to show that this does not hold for a general n. In other words show that $D(n^k) = kn^{k-1}$ does not hold.

   Solution to Problem 4:

   (a) $D(899) = D(29 \cdot 31) = 29D(31) + 31D(29) = 60$.

   (b) $D(36) = 4D(9) + 9D(4) = 4(2 \cdot 3) + 9(4) = 24 + 36 = 60$.

   (c) We see that $D(36) = 60$ which is not $2 \cdot 6 = 12$, so that is a valid counterexample.

5. **[6 pts]** Let $n = p_1^{e_1} p_2^{e_2} ... p_k^{e_k}$ be the prime factorization of n. Prove that $D(n) = n \sum_{i=1}^{k} \frac{e_i}{p_i}$.

   Solution to Problem 5:

   We can write $n = \prod_{i=1}^{i=t} p_i$ for primes p_i that are not necessarily distinct. This is essentially expanding the prime factorization out to individual prime terms. Now, the theorem simplifies to proving that $D(n) = n \sum_{i=1}^{t} \frac{1}{p_i}$. We will prove this theorem by induction on the number of primes t.

   Our base case is $t = 0$, in which case n is the empty product which is 1. Therefore, we want to show that $D(n) = \sum_{i=1}^{t} \frac{1}{p_i}$. Since $t = 0$, this is simply the empty sum which is also 0. We have previously shown that $1' = 0$ so we are done.

   The induction step is when $i = k - 1$ for some k, so $n = \prod_{i=1}^{i=k-1} p_i$. Then, $D(n) = n \sum_{i=1}^{k-1} \frac{1}{p_i}$. Now we consider np_k. We see that $D(np_k) = D(n)p_k + nD(p_k) = np_k \sum_{i=1}^{k-1} \frac{1}{p_i} + n = np_k(\sum_{i=1}^{k-1} \frac{1}{p_i} + \frac{1}{p_k}) = np_k(\sum_{i=1}^{k} \frac{1}{p_i})$ and we are done.

6. Now that we have proved a general formula, let's go back and reexamine the power rule.

   (a) **[1 pt]** Compute $D(720)$

   (b) **[1 pt]** Compute $D(12^3)$

   (c) **[1 pt]** Compute $D(14^5)$. (*Hint*: $14^4 = 38416$).

   (d) **[3 pts]** Find and prove a formula for $D(n^k)$ in terms of $D(n)$, n, and k.

   Solution to Problem 6:

   (a) The prime factorization of 720 is $3^2 * 2^4 * 5$. So our summation turns out to be $720 \cdot (\frac{2}{3} + \frac{4}{2} + 15) = 720 \cdot \frac{43}{15} = \boxed{2064}$.

   (b) We see $12^3 = 2^6 \cdot 3^3$. So our formula states that this is $12^3 \cdot (\frac{6}{2} + \frac{3}{3}) = 12^3 \cdot 4 = 6912$.

   (c) We see that $14^5 = 7^5 \cdot 2^5$. So we have $14^5 \cdot (\frac{5}{2} + \frac{5}{7}) = 14^5 \cdot \frac{45}{14} = 14^4 \cdot 45 = 38416 \cdot 45 = \boxed{1728720}$.

   (d) We will show that $D(n^k) = kn^{k-1}D(n)$. First, let $n = p_1^{e_t} ... p_n^{e_t}$ where $p_1 p_2 ... p_t$ are distinct primes. Then $n^k = p_1^{ke_t} ... p_n^{ke_t}$ by exponent rules. Therefore $D(n^k) = n^k \cdot \sum_{i=1}^{t} \frac{ke_t}{p_t} = kn^k \cdot \sum_{i=1}^{t} \frac{e_t}{p_t} = kn^{k-1}(n \sum_{i=1}^{t} \frac{e_t}{p_t}) = kn^{k-1}D(n)$.

Remark. You may notice some striking similarities to the chain rule in calculus. That is $\frac{d}{dx}f(x)^k = kf(x)^{k-1} \cdot \frac{df}{dx}$.

7. **The Goldbach Conjecture** is a famous conjecture in mathematics that states that for any even integer $2k > 2$, there exists two primes p, q such that $p + q = 2k$.

   [4 pt] Consider the equation $D(n) = 2k$. Show that if there exists a $k \in \mathbb{N}$, greater than 1 such that $D(n) \neq 2k$, for all $n \in \mathbb{N}$, then the Goldbach Conjecture is false.

   Solution to Problem 7: We consider the contrapositive of the statement. That is we will show that if the Goldbach Conjecture is true, then there exists a solution for $D(n) = 2k$. If the Goldbach Conjecture is true, then there exists p, q such that $p + q = 2k$. Consider now $n = pq$. Taking the arithmetic derivative, we see that $D(n) = pD(q) + qD(p)$. Since p, q are prime, this is simply $p + q$. So, if the Goldbach Conjecture is true, then we can construct an n such that $D(n) = 2k$ for any k.

Higher Order Arithmetic Derivatives

We have explored the notion of the arithmetic derivative. Now, let us see what happens as we iterate this function.

Definition. Define the kth order derivative, $D^{(k)}(n)$ of a natural number n as:

$$\begin{cases} D(n) & \text{if } k = 1. \\ D(D^{(k-1)}(n)) & \text{if } k > 1 \end{cases}$$

Example. As an example, we will compute $D^{(3)}(21)$.

We see that $D^{(3)}(21) = D(D(D(21)))$. $D(21) = 21 \cdot (\frac{1}{7} + \frac{1}{3}) = 10$. $D(10) = 10(\frac{1}{5} + \frac{1}{2}) = 7$. Finally since 7 is prime, $D(7) = 1$, and we are done.

8. Compute the following arithmetic derivatives.

   (a) **[1 pt]** $D^{(2)}(34)$

   (b) **[1 pt]** $D^{(3)}(49)$

   (c) **[1 pt]** $D^{(4)}(3125)$

   (d) **[1 pt]** $D^{(4)}(64)$

   Solution to Problem 8:

   (a) $D(D(34)) = D(17 \cdot D(2) + D(17) \cdot D(2)) = D(17 + 2) = D(19) = \boxed{1}$.

   (b) First, $D(49) = D(7^2) = 2 \cdot 7 = 14$, by our proof of prime powers. Then $D(14) = D(2 \cdot D(7) + 7 \cdot D(2)) = D(9) = \boxed{6}$.

   (c) We see that $3125 = 5^5$ $D(5^5) = 5 \cdot D(5^4) = 5^5$. So $D^{(n)}(5^5) = 5^5 = \boxed{3125}$.

   (d) $D(64) = D(2^6) = 6 \cdot 2^5 = 3 \cdot 2^6$. $D(3 \cdot 2^6) = 3 \cdot D(2^6) + 2^6 \cdot D(3) = 9 \cdot 2^6 + 2^6) = 10 \cdot 2^6 . D(10 \cdot 2^6) = D(10) \cdot 2^6 + 10 \cdot D(2^6) = 7 \cdot 2^6 + 30 \cdot 2^6 = 37 \cdot 2^6$. Finally, $D(37 \cdot 2^6) = 37 \cdot D(2^6) + 2^6 \cdot D(37) = 112 \cdot 2^6 = 7168$.

9. In this problem, we examine the case in which $D(n) = n$. This is known as an *arithmetic differential equation.*

   (a) **[2 pt]** Show that if $n = p^p$ for a prime p, then $D(n) = n$.

   (b) **[5 pts]** Prove that the numbers $n = p^p$ for any prime p are the only solutions to $D(n) = n$.

   Solution to Problem 9:

   (a) By our prime powers formula, we have $D(p^p) = p \cdot p^{p-1} = p^p$ and we are done.

   (b) Suppose $D(n) = n$. Then, if $p_1^{e_1} p_2^{e_2} ... p_k^{e_k}$ is the prime factorization of n, by our summation formula, $n \cdot \sum_{i=1}^{k} \frac{e_i}{p_i} = n$. This implies that $\sum_{i=1}^{k} \frac{e_i}{p_i} = 1$. Multiplying by $\prod_{i=1}^{k} p_i$, we have $\prod_{i=1}^{k} p_i = \sum_{i=1}^{k} \frac{e_i}{p_i} \prod_{i=1}^{k} p_i$. Bringing, all but the first term to the left hand side, we have $\prod_{i=1}^{k} p_i (1 - \sum_{i=2}^{k} \frac{e_i}{p_i}) = \prod_{i=2}^{k} p_i e_i$. The left hand side divides p_1, so therefore both sides must as well. Since $p_2 ... p_k$ are relatively prime to p_1, $e_i = g p_1$ for some $g \geq 1$. So we know that $\sum_{i=2}^{k} \frac{e_i}{p_i} + g = 1$. Since $g \geq 1$ and all terms in the summation are nonnegative, we see that $g = 1$ and $\sum_{i=2}^{k} \frac{e_i}{p_i} = 0$, implying that $n = p_1^{p_1}$, proving the desired statement.

Sequences of Higher Order Derivatives

Let us now look at sequence of derivatives. That is, we will consider the sequence of the higher order derivatives of n as $\Delta(n)$. Specifically, we will define $(\Delta(n))_k$ as the sequence

$$D^{(1)}(n), D^{(2)}(n), D^{(3)}(n), D^{(4)}(n)\ldots$$

where the kth term in the sequence is $D^{(k)}(n)$.

Definition. Sequences like these can either be increasing, decreasing, or neither. We call a sequence $(s)_n$ **increasing** if $s_{n+1} \geq s_n$ for all $n \in \mathbb{N}$. A sequence is **decreasing** if $s_{n+1} \leq s_n$ for all $n \in \mathbb{N}$.

10. (a) **[1 pt]** Is the sequence, $(\Delta(12))$, increasing, decreasing, or neither? You do not need to justify your answer.

    (b) **[1 pt]** Is the sequence, $(\Delta(14))$, increasing, decreasing, or neither? You do not need to justify your answer.

    (c) **[2 pts]** Find an example of a number such that $(\Delta(n))$ is neither increasing nor decreasing. You must prove your answer.

    (d) **[2 pts]** For which k is the sequence $(\Delta(2^k))$, increasing? Your answer should be a condition on k. You must prove your answer.

    Solution to Problem 10:

    (a) The sequence is increasing.

    (b) The sequence is decreasing.

    (c) Starting with 15, we see that $D(15) = 8$. And $D(8) = 12$. So we see that $(\Delta(15))$ decreases, and then increases, so it it is neither increasing nor decreasing.

    (d) We will show that $(\Delta(2^k))$ is increasing if $k \geq 2$. Consider $a = c \cdot 2^2$ Then, $D(a) = D(c) \cdot 2^2 + D(2^2) \cdot c$. Setting $D(c) + c = g$, we see that $D(a) = g \cdot 2^2 \geq c \cdot 2^k$. Considering the sequence $\Delta(2^k)$, the first term s_1 and all subsequent terms s_n can be written as $c \cdot 2^2$. As a result, each term $s_{n+1} \geq s_n$ for all n and we are done. Now, if $k = 1$, we already see that $D(2) = 1$ so it is not increasing. Now, if $k = 0$, then $D(1) = 0$, so the sequence simply becomes all 0's, which is increasing as defined by our defintiion. Therefore, the set $S = \{k \text{ if } \Delta(2^k) \text{ is increasing}\}$ is $k \geq 2$ and $k = 0$. *Note: During the Power Round, we did not intend for $k = 0$ to be a solution, so as long as $k \geq 2$, the answer was accepted.*

11. **[3 pts]** Show that if $n = k \cdot p^p$ for some natural number $k > 1$ and prime p, then $(\Delta(n))$ is *strictly increasing*, meaning $s_{n+1} > s_n$ for all n.

    Solution to Problem 11:

    Consider $a = k \cdot p^p$ Then, $D(a) = D(k) \cdot p^p + D(p^p) \cdot k = (D(k) + k)p^p$. Setting $D(k) + k = g$, we see that $D(a) = g \cdot p^p > k \cdot p^p$, since $k > 1$. Considering the sequence $\Delta(k \cdot p^p)$, the first term s_1 and all subsequent terms s_n can be written as $k_n \cdot p^p$ for some sequence (k_n). Therefore, each term $s_{n+1} > s_n$ for all n and we are done.

12. **[7 pts]** Suppose $(\Delta(n))$ is such that it alternates between two distinct numbers m and n. Show that $gcd(n, m) = 1$ and neither m nor n are divisible by a square number other than 1.

    Solution to Problem 12:

If $(\Delta(n))$ consists alternates between two numbers m and n, then this implies that $D(n) = m$ and $D(m) = n$, with $m \neq n$. First neither m or n, are divisible by p^p for any prime p, since we already showed that such a sequence is strictly increasing in the last problem, so it cannot alternate.

Lemma. *Let p^k be the highest power of p that divides n such that $0 < k < p$. Then p^{k-1} divides is the largest power of p that divides $D(n)$.*

Proof. Let $n = p^k \cdot m$ for some m. Then, $D(n) = kp^{k-1}m + D(m)p^k = p^{k-1}(km + D(m)p)$. Since $k < p$ and $gcd(m, p) = 1$, we see that km does not divide p so the expression $km+D(m)p$ does not divide p, so p^{k-1} is the largest power of p that divides $D(n)$.

□

Let $gcd(n, m) = d$. Let p be the smallest prime p that divides d and k be the largest prime power of p such that p^k divides m. Then, $k < p$ since otherwise $p^p|m$ which implies that $p^p|m, n$ which is not possible. So, therefore $D(D(m)) = m$ has the highest power p^{k-2} a contradiction since k is the highest power of p such that p^k divides m. As a result, $gcd(n, m) = 1$.

Now, if m is divisible by a square s^2. Then, let q be the smallest prime factor of s. Then $q^2|m$. Clearly, q cannot be 2 since otherwise m is divisible by p^p, so $2 < q$. By the above lemma, $D(m) = n$ has q as a factor, but then $gcd(m, n) \geq q$, a contradiction. So m is square-free. Since m and n are symmetric, since the kth term of $(\Delta(n))$ is the $k+1$st term of $(\Delta(m))$ we see that n is also square-free, or not divisible by a square number other than 1.

Bounds on the Arithmetic Derivative

Let's try to understand what bounds the arithmetic derivative.

13. **[4 pts]** Let p^* be the smallest prime factor of n. Show that $D(n) \leq \frac{n \log_{p^*}(n)}{p^*}$.

    Solution to Problem 13: Let $n = p_1 p_2 ... p_k$ for $p_i \geq p^*$ where the p_i's are not necessarily distinct. By our summation formula, $D(n) = n \sum_{i=1}^{k} \frac{1}{p_i} \leq n \sum_{i=1}^{k} \frac{1}{p^*} = \frac{nk}{p^*}$. Since $n = p_1 p_2 ... p_k \geq (p^*)^k$, we see that $k \leq \log_{p^*}(n)$ so $D(n) \leq \frac{nk}{p^*} \leq \frac{n \log_{p^*}(n)}{p^*}$.

14. **[4 pts]** Let k be the sum of all the exponent values in the prime factorization of n. Show that $D(n) \geq k \cdot n^{1-\frac{1}{k}}$

    Solution to Problem 14:

    We see that $n = p_1 ... p_k$ where p_i's are not necessarily distinct. So we have $\frac{D(n)}{n} = \sum_{i=1}^{k} \frac{1}{q_i}$. By the $AM - GM$ inequality, $\sum_{i=1}^{k} \frac{1}{q_i} \geq k \cdot (\prod_{i=1}^{k} \frac{1}{q_i})^{\frac{1}{k}}) = kn^{\frac{-1}{k}}$. So $D(n) \geq kn^{1-\frac{1}{k}}$.

15. **[6 pts]** Let k be the sum of all the exponent values in the prime factorization of n. Show that $D(n) \leq \frac{k-1}{2}n + 2^{k-1}$

    Solution to Problem 15:

    First we note that if $k = 1$, then n is a prime and this is trivially true since $\frac{k-1}{2}n + 2^{k-1} = 1$. Now, n can be expressed as $p_1 ... p_k$ where p_i's are not necessarily distinct. So we assume that $k > 1$.

Case 1: Suppose n has at least three factors that are greater than or equal to 3. Then, $D(n) = n\sum_{i=1}^{k}\frac{1}{p_i} \le n\sum_{i=1}^{k-3}\frac{1}{2} + 3\frac{1}{3} = n\frac{k-1}{2} \le \frac{k-1}{2}n + 2^{k-1}$ and we are done.

Case 2: If n has at most one factor that is greater than or equal to 3. Then, $n = 2^{k-1}p$ for some prime p. Then $D(n) = \frac{n(k-1)}{2} + \frac{n}{p} = \frac{k-1}{2}n + 2^{k-1}$ and we are done.

Case 3: If n has two factors that are greater than or equal to 3. Then, $n = 2^{k-2}p_1p_2$ for some primes p_1, p_2. Then $D(n) = n(\frac{k-2}{2} + \frac{1}{p_1} + \frac{1}{p_2})$. We see that $2p_1 + 2p_2 \le p_1p_2 + 2$, since $p_1p_2 - 2p_1 - 2p_2 + 4 = (p_1 - 2)(p_2 - 2) \ge 0$, since $p_1, p_2 \ge 3$. Dividing by p_1p_2 we have $\frac{1}{p_1} + \frac{1}{p_2} \le \frac{1}{2} + \frac{2}{p_1p_2}$, so we see that $D(n) \le n(\frac{(k-2)}{2} + \frac{1}{2} + \frac{2}{p_1p_2}) \le n\frac{k-1}{2} + \frac{2n}{p_1p_2} = n\frac{k-1}{2} + 2^{k-1}$ and we are done.

These cases cover all possible values of n so we are done.