1. Find the sum of the squares of all values of x that satisfy

$$\log_2(x+3) + \log_2(2-x) = 2.$$

Answer: 5

Solution: We use the sum rule for logs to get

$$\log_2(-x^2 - x + 6) = 2.$$

We raise 2 to the power of both sides of the equation, giving us

$$-x^2 - x + 6 = 4.$$

Subtracting 4 from both sides gives us

$$-x^2 - x + 2 = (-x+1)(x+2) = 0 \implies x = -2, 1$$

which both satisfy the original equation (in both cases, $x+3 > 0$ and $2-x > 0$, so the logarithms are defined) giving an answer of $(-2)^2 + 1^2 = \boxed{5}$.

2. The polynomial $f(x) = x^3 + rx^2 + sx + t$ has r, s, and t as its roots (with multiplicity), where $f(1)$ is rational and $t \neq 0$. Compute $|f(0)|$.

Answer: 1

Solution: First, we have by Vieta's formulae that $rst = -t$. Since $t \neq 0$, $rs = -1$, so we write

$$s = -\frac{1}{r}.$$

Now we also observe (from Vieta's formulae) that $r + s + t = -r$, so $t = -2r - s = -2r + \frac{1}{r}$. Now we can write

$$\begin{aligned} f(x) &= (x-r)(x-s)(x-t) \\ &= (x-r)\left(x + \frac{1}{r}\right)\left(x + 2r - \frac{1}{r}\right) \\ &= x^3 + rx^2 - \left(2r^2 - 2 + \frac{1}{r^2}\right)x - 2r + \frac{1}{r} \\ &= x^3 + rx^2 + sx + t \\ &= x^3 + rx^2 - \frac{1}{r}x - 2r + \frac{1}{r}. \end{aligned}$$

Equating the coefficients of x in the third and fifth lines above yields $\frac{1}{r} = 2r^2 - 2 + \frac{1}{r^2}$, so $2r^4 - 2r^2 - r + 1 = 0$. We are given that

$$f(1) = r + s + t + 1 = -r + 1$$

is rational, so r must be rational. By the rational root theorem, the only possible values for r are ± 1 and $\pm\frac{1}{2}$. A simple check reveals that $r = 1$ is the only possibility, whence we find

$$f(x) = x^3 + x^2 - x - 1,$$

so $|f(0)| = \boxed{1}$.

3. Let x and y be integers from -10 to 10, inclusive, with $xy \neq 1$. Compute the number of ordered pairs (x, y) such that $\left|\frac{x+y}{1-xy}\right| \leq 1$.

Answer: 365

Solution: Either

$$-1 \leq \frac{x+y}{1-xy} \implies x+y \geq xy-1 \implies xy-x-y-1 \leq 0 \implies (x-1)(y-1) \leq 2$$

by SFFT, or similarly,

$$x+y \leq 1-xy \implies xy+x+y+1 \leq 2 \implies (x+1)(y+1) \leq 2.$$

Both of the boundaries of these graphs are hyperbolae, so we can observe that either $(x, y) = (\pm 1, 0), (0, \pm 1)$, or $(0, 0)$, or one of the variables (but not both) is negative and the other is positive, giving $5 + 2 \cdot 10^2 = 205$ solutions for $|x|, |y| \leq 10$.

However, this is only for the case where x and y have opposite signs. In multiplying through by $1 - xy$, we lose those cases in which x and y the same sign, as this makes $1 - xy$ negative. Indeed, we have those pairs in the square with vertices at $(2, 2), (2, 10), (10, 10), (10, 2)$ and those pairs in the square with vertices $(-2, -2), (-2, -10), (-10, -10), (-10, -2)$ with the exception of the pairs $(2, 2)$ and $(-2, -2)$, for an additional $2 \cdot 9^2 - 2 = 160$ ordered pairs. In total, there are $205 + 160 = \boxed{365}$ satisfactory ordered pairs (x, y).