1. What is the sum of all two-digit odd numbers whose digits are all greater than 6?

   Answer: 528

   Solution: The odd numbers made up with the digits 7, 8 and 9 are $77, 79, 87, 89, 97$ and 99. These can be summed up manually, or we note that the average of these numbers is 88, so the sum is $88 \cdot 6 = \boxed{528}$.

2. Define an operation $\diamondsuit$ as $a \diamondsuit b = 12a - 10b$. Compute the value of $((((20 \diamondsuit 22) \diamondsuit 22) \diamondsuit 22) \diamondsuit 22)$.

   Answer: 20

   Solution: We compute $20 \diamondsuit 22 = 12(20) - 10(22) = 20$. Thus, we can replace every instance of $20 \diamondsuit 22$ with 20:

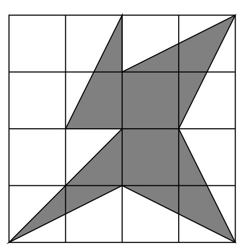
$$\begin{aligned}((((20 \diamondsuit 22) \diamondsuit 22) \diamondsuit 22) \diamondsuit 22) &= (((20 \diamondsuit 22) \diamondsuit 22) \diamondsuit 22) \\ &= ((20 \diamondsuit 22) \diamondsuit 22) \\ &= 20 \diamondsuit 22 \\ &= \boxed{20}.\end{aligned}$$

3. For lunch, Lamy, Botan, Nene, and Polka each choose one of three options: a hot dog, a slice of pizza, or a hamburger. Lamy and Botan choose different items, and Nene and Polka choose the same item. In how many ways could they choose their items?

   Answer: 18

   Solution: There are 3 ways for Lamy to choose an item. After that, there are 2 ways for Botan to choose a different item from Lamy. Then there are 3 ways for Nene to choose an item, and after that there is just 1 way for Polka to choose the same item as Nene. The number of ways for them to choose the items is $3 \cdot 2 \cdot 3 \cdot 1 = \boxed{18}$.

4. Big Chungus has been thinking of a new symbol for BMT, and the drawing below is what he came up with. If each of the 16 small squares in the grid are unit squares, what is the area of the shaded region?



   Answer: 6

   Solution: We can divide the region into multiple parts by considering each of 2 by 2 sections in the grid:

   - Top left: The shaded area in this section is a right triangle with legs of length 1 and 2, and its area is $\frac{1}{2} \cdot 1 \cdot 2 = 1$.
   - Top right: The unshaded area in this section is two right triangles with legs of length 1 and 2, so the area of the unshaded region is $4 - 1 - 1 = 2$.

- Bottom left: The shaded area in this section is a triangle with a (vertical) base of length 1 and a (horizontal) height of length 2, and its area is $\frac{1}{2} \cdot 1 \cdot 2 = 1$.
- Bottom right: The unshaded area in this section is two right triangles with legs of length 1 and 2, so the area of the unshaded region is $4 - 1 - 1 = 2$.

The total area of the shaded region is $1 + 2 + 1 + 2 = \boxed{6}$.

5. Compute the last digit of $(5^{20} + 2)^3$.

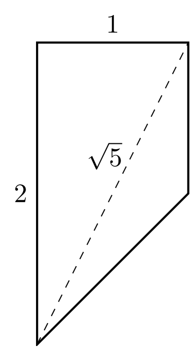
   Answer: 3

   Solution: To find the last digit of an expression, we only need to keep track of what the last digit of each number as we compute the expression. The last digit of 5^{20} is 5 since 5 multiplied by itself any number of times always ends in 5. Hence, $5^{20} + 2$ ends in a 7. The last digit of $(5^{20} + 2)^3$ then has the same last digit of $7^3 = 343$, so the last digit is $\boxed{3}$.

6. To fold a paper airplane, Austin starts with a square paper $FOLD$ with side length 2. First, he folds corners L and D to the square's center. Then, he folds corner F to corner O. What is the longest distance between two corners of the *resulting* figure?

   Answer: $\sqrt{5}$

   Solution:



After folding, the resulting figure will look like the diagram above. The two corners which are the furthest from each other are the top-right and bottom-left corners. To find this length, we use the Pythagorean theorem: $\sqrt{1^2 + 2^2} = \boxed{\sqrt{5}}$.

7. Let $f(x) = x^2 + \lfloor x \rfloor^2 - 2x \lfloor x \rfloor + 1$. Compute $f(4 + \frac{5}{6})$. (Here, $\lfloor m \rfloor$ is defined as the greatest integer less than or equal to m. For example, $\lfloor 3 \rfloor = 3$ and $\lfloor -4.25 \rfloor = -5$.)

   Answer: $\frac{61}{36}$ or $1\frac{25}{36}$

   Solution: Note that

$$f(x) = (x - \lfloor x \rfloor)^2 + 1 = \{x\}^2 + 1$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x. It follows that

$$f\left(4 + \frac{5}{6}\right) = \left\{4 + \frac{5}{6}\right\}^2 + 1 = \left(\frac{5}{6}\right)^2 + 1 = \boxed{\frac{61}{36}}.$$

8. Oliver is at a carnival. He is offered to play a game where he rolls a fair dice and receives \$1 if his roll is a 1 or 2, receives \$2 if his roll is a 3 or 4, and receives \$3 if his roll is a 5 or 6. Oliver

plays the game repeatedly until he has received a total of at least \$2. What is the probability that he ends with \$3?

Answer: $\frac{4}{9}$

Solution: In order for Oliver to receive exactly \$3, he must roll a 5 or 6 on the first roll, or roll a 1 or 2 followed by rolling a 3 or 4. This occurs with probability $\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3}=\boxed{\frac{4}{9}}$.

9. What is the measure of the largest convex angle formed by the hour and minute hands of a clock between 1:45 PM and 2:40 PM, in degrees? Convex angles always have a measure of less than 180 degrees.

Answer: 160

Solution: At 1:45, the measure of the convex angle is $4(30)+\frac{3}{4}(30)=142.5^\circ$. At 2:40, the measure of the convex angle is $180-\frac{2}{3}(30)=160^\circ$. The angles in between are all less than 160°: they decrease from 1:45 until the hour hand and minute hand meet at a time between 2:10 and 2:15, then increase up until 2:45. So the largest measure that occurs is $\boxed{160}^\circ$.

10. Each box in the equation

$$\square\times\square\times\square-\square\times\square\times\square=9$$

is filled in with a different number in the list 2, 3, 4, 5, 6, 7, 8 so that the equation is true. Which number in the list is not used to fill in a box?

Answer: 4

Solution: Since the difference between the products is an odd number, one of the products must be odd, and the three boxes in the odd product must all contain odd numbers. The only three odd numbers in the list are 3, 5 and 7, so they must be used in the odd product. Therefore, one of the products must be $3\times5\times7=105$. Then depending which product in the equation is odd, the other product is either $105-9=96$ or $105+9=114$.

One way to finish is to try combinations of products of 2, 4, 6 and 8 to see if 96 or 114 are attainable. A quicker way is to note that the missing number is either equal to $\frac{2\times4\times6\times8}{96}=4$ or $\frac{2\times4\times6\times8}{114}=\frac{64}{19}$. Therefore, the number that is not used must be $\boxed{4}$. (To check our answer, we compute $3\times5\times7-2\times6\times8=105-96=9$, so this works.)

11. The equation

$$4^x-5\cdot2^{x+1}+16=0$$

has two integer solutions for x. Find their sum.

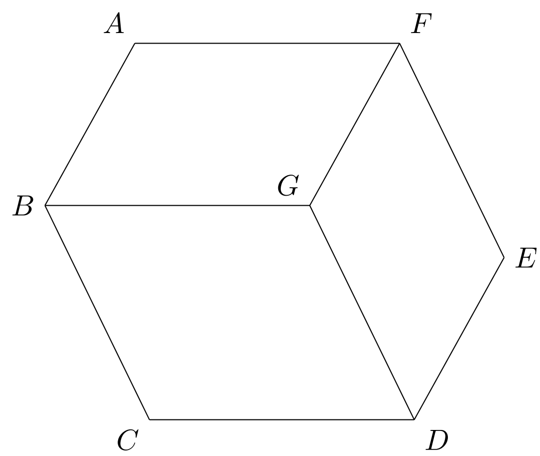
Answer: 4

Solution: Define $y=2^x$. Then our equation can be rewritten as

$$y^2-10y+16=(y-2)(y-8)=0.$$

This gives $y=2$ or $y=8$, which means $x=1$ or $x=3$. Therefore, the sum of the solutions for x is $\boxed{4}$.

12. Parallelograms $ABGF$, $CDGB$ and $EFGD$ are drawn so that $ABCDEF$ is a convex hexagon, as shown. If $\angle ABG=53^\circ$ and $\angle CDG=56^\circ$, what is the measure of $\angle EFG$, in degrees?



Answer: 71

Solution: The angles around G must sum to 360°. Using the parallelograms, we have $\angle FGB = 180^\circ - 53^\circ = 127^\circ$ and $\angle BGD = 180^\circ - 56^\circ = 124^\circ$, so $\angle DGF = 360^\circ - 127^\circ - 124^\circ = 109^\circ$. Then $\angle EFG = 180^\circ - 109^\circ = \boxed{71}^\circ$.

13. Three standard six-sided dice are rolled. What is the probability that the product of the values on the top faces of the three dice is a perfect cube?

Answer: $\mathbf{\frac{1}{18}}$

Solution: Let the numbers rolled be a, b, and c. We want to check when their product abc is a perfect cube. Notice that the maximum cube that could result from the product of the top faces is $6 \cdot 6 \cdot 6 = 6^3$. Casework on abc:

- $abc = 1^3$: If any of a, b, c are greater than 1, then $abc > 1$. The only possibility is $(1, 1, 1)$.
- $abc = 2^3$: a, b, and c must be powers of 2, and in particular must be in $\{1, 2, 4\}$. By inspection, $(2, 2, 2)$ and every permutation of $(1, 2, 4)$ works, for a total of $1 + 6 = 7$ cases.
- $abc = 3^3$: $a, b, c \in \{1, 3\}$, so the only possibility is $(3, 3, 3)$.
- $abc = 4^3$: $a, b, c \in \{1, 2, 4\}$, so the only possibility is $(4, 4, 4)$.
- $abc = 5^3$: $a, b, c \in \{1, 5\}$, so the only possibility is $(5, 5, 5)$.
- $abc = 6^3$: If any of a, b, c are less than 6, then $abc < 6^3$. The only possibility is $(6, 6, 6)$.

There are 12 such ordered triples, and the answer is $\frac{12}{6^3} = \boxed{\frac{1}{18}}$.

14. Compute the number of positive integer divisors of 100000 which do not contain the digit 0.

Answer: 11

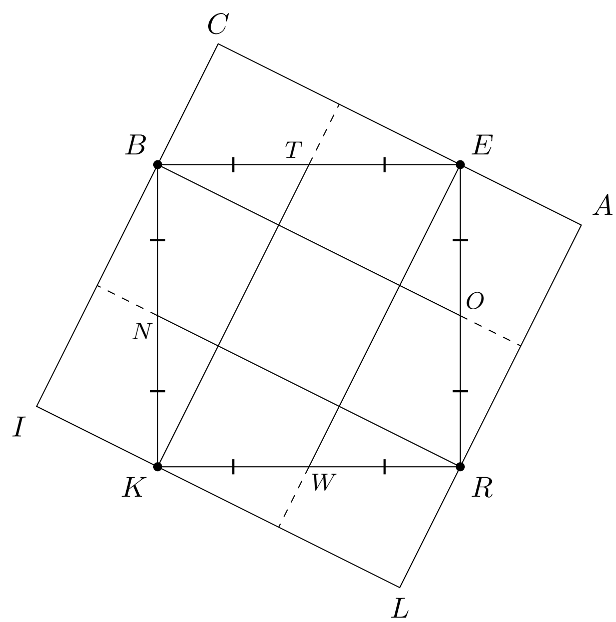
Solution: Note that $100000 = 2^5 \cdot 5^5$. Any multiple of 10 ends in a 0, so a divisor of $2^5 \cdot 5^5$ that does not contain a 0 is either not divisible by 2 or not divisible by 5. We consider the cases separately.

- A divisor of $2^5 \cdot 5^5$ that is not divisible by 5 will not contain the prime factor 5, and thus it is either 1 or only contains the prime factor 2, so it must be a power of 2. The powers of 2 up to 2^5 are 1, 2, 4, 8, 16, and 32.

- A divisor of $2^5 \cdot 5^5$ that is not divisible by 2 will not contain the prime factor 2, so it is either 1 or only contains the prime factor 5, and thus it must be a power of 5. The powers of 5 up to 5^5 are 1, 5, 25, 125, 625, and 3125.

Since 1 appears in both lists, the total number of divisors of $2^5 \cdot 5^5$ that do not contain a 0 is $6 + 6 - 1 = \boxed{11}$.

15. Sohom constructs a square $BERK$ of side length 10. Darlnim adds points T, O, W, and N, which are the midpoints of $\overline{BE}$, $\overline{ER}$, $\overline{RK}$, and $\overline{KB}$, respectively. Lastly, Sylvia constructs square $CALI$ whose edges contain the vertices of $BERK$, such that $\overline{CA}$ is parallel to $\overline{BO}$. Compute the area of $CALI$.



Answer: 180

Solution: Note that

$$[CALI] = [BERK] + [\triangle EAR] + [\triangle RLK] + [\triangle KIB] + [\triangle BCE] = [BERK] + 4[\triangle EAR].$$

We know $[BERK] = 10^2 = 100$, so it remains to calculate $[EAR]$. Let $\overline{EW}$ and $\overline{RN}$ intersect at X. Since $\angle EAR = 90^\circ$ and $\overline{EA} \parallel \overline{XR}$ and $\overline{EX} \parallel \overline{AR}$, we can conclude that $EXRA$ is a rectangle. Thus, $[\triangle EAR] = [\triangle EXR]$.

Now, note that since $\angle EXR = \angle ERW = 90^\circ$, we have that $\triangle EXR \sim \triangle ERW$. Hence

$$\frac{XR}{ER} = \frac{RW}{EW} \implies \frac{XR}{10} = \frac{5}{\sqrt{5^2 + 10^2}},$$

from which we get $XR = 2\sqrt{5}$. Hence $[\triangle EXR] = \frac{1}{2} \cdot XR \cdot (2 \cdot XR) = 20$, so the answer is $100 + 4(20) = \boxed{180}$.

16. A street on Stanford can be modeled by a number line. Four Stanford students, located at positions 1, 9, 25 and 49 along the line, want to take an UberXL to Berkeley, but are not sure where to meet the driver. Find the smallest possible total distance walked by the students to a

single position on the street. (For example, if they were to meet at position 46, then the total distance walked by the students would be $45 + 37 + 21 + 3 = 106$, where the distances walked by the students at positions 1, 9, 25 and 49 are summed in that order.)

Answer: 64

Solution: The total distance walked to a given position x is equal to:

$$f(x) = |x - 1| + |x - 9| + |x - 25| + |x - 49|$$

We observe the behavior of this function along intervals to find the minimum value:

- If $x < 1$ or $x > 49$, then $x = 1$ or $x = 49$ respectively would decrease all distances, so the total distance would also decrease.
- When $1 \leq x \leq 9$, $f(x) = (x-1)+(9-x)+(25-x)+(49-x) = 82-2x$, and the minimal value on this interval occurs at $x = 9$, where $f(9) = 82 - 2(9) = 64$.
- When $9 \leq x \leq 25$, $f(x) = (x-1) + (x-9) + (25-x) + (49-x) = 64$.
- When $25 \leq x \leq 49$, $f(x) = (x-1) + (x-9) + (x-25) + (49-x) = 2x + 14$, and the minimal value on this interval occurs at $x = 25$, where $f(25) = 2(25) + 14 = 64$.

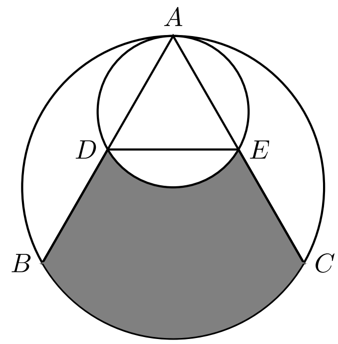
We conclude that the smallest possible value of the sum of distances, or the minimum value of $f(x)$, is $\boxed{64}$.

17. Midori and Momoi are arguing over chores. Each of 5 chores may either be done by Midori, done by Momoi, or put off for tomorrow. Today, each of them must complete at least one chore, and more than half of the chores must be completed. How many ways can they assign chores for today? (The order in which chores are completed does not matter.)

Answer: 160

Solution: We use complementary counting. For all assignments without restrictions, each chore may be done by Midori, Momoi, or put off to tomorrow. Without the restrictions, there are $3^5 = 243$ ways to assign the chores. The assignments that violate the restrictions either have Midori or Momoi not doing any chores, or each person doing exactly one chore. For the first case, there are 2^5 assignments where Midori doesn't do any chores, 2^5 assignments where Momoi doesn't do any chores, and 1 overlapping assignment where both of them don't do any chores, for a total of $2^5 + 2^5 - 1 = 63$ such assignments. For the second case, there are 5 ways to pick a chore for Midori and 4 ways to pick a different chore for Momoi, for a total of $5 \cdot 4 = 20$ such assignments. The desired number of assignments of chores is $243 - 63 - 20 = \boxed{160}$.

18. Let equilateral triangle $\triangle ABC$ be inscribed in a circle ω_1 with radius 4. Consider another circle ω_2 with radius 2 internally tangent to ω_1 at A. Let ω_2 intersect sides $\overline{AB}$ and $\overline{AC}$ at D and E, respectively, as shown in the diagram. Compute the area of the shaded region.



Answer: $\mathbf{6\sqrt{3} + 4\pi}$

Solution: Let O_1 and O_2 be the centers of the circles ω_1 and ω_2, respectively. If the foot of the altitude from A to $\overline{DE}$ is F, then we have that $\triangle ADF$ and $\triangle DO_2F$ are both 30-60-90 triangles. Hence $AF = DF\sqrt{3}$ and $O_2F = DF/\sqrt{3}$, so since $AO_2 = 2$ we can compute $DF = \sqrt{3}$, so $AD = AE = 2\sqrt{3}$.

Now, let R_1 be the region bounded by $\overline{AB}$, $\widehat{BC}$, $\overline{CA}$, and let R_2 be the region bounded by $\overline{AD}$, $\widehat{DE}$, and $\overline{EA}$. Since $\triangle ADE \sim \triangle ABC$, the region R_1 is equivalent to the region R_2 dilated from A with ratio $AO_1/AO_2 = 2$. Hence the ratio of their areas is $2^2 = 4$. So, the desired answer is $[R_1] - [R_2] = 3[R_2]$.

Note that $[R_2] = [\triangle ADO_2] + [\triangle AEO_2] + \frac{1}{3}[\omega_2]$, since $\angle DO_2E = 120°$. This can be computed using the lengths from before as $\sqrt{3} + \sqrt{3} + \frac{4\pi}{3}$, so the answer is $3\left(2\sqrt{3} + \frac{4\pi}{3}\right) = \boxed{6\sqrt{3} + 4\pi}$.

19. Suppose we have four real numbers a, b, c, d such that a is nonzero, a, b, c form a geometric sequence, in that order, and b, c, d form an arithmetic sequence, in that order. Compute the smallest possible value of $\frac{d}{a}$. (A geometric sequence is one where every succeeding term can be written as the previous term multiplied by a constant, and an arithmetic sequence is one where every succeeeding term can be written as the previous term added to a constant.)

Answer: $\mathbf{-\frac{1}{8}}$

Solution: Let r be the ratio in the geometric sequence, so that $b = ar$ and $c = ar^2$. Since $d - c = c - b$, we have $d = 2c - b = a \cdot (2r^2 - r)$. The minimum of $\frac{d}{a} = 2r^2 - r$ occurs at $r = -\frac{-1}{2 \cdot 2} = \frac{1}{4}$, with value $2 \cdot \left(\frac{1}{4}\right)^2 - \frac{1}{4} = \boxed{-\frac{1}{8}}$.

20. The game Boddle uses eight cards numbered $6, 11, 12, 14, 24, 47, 54$, and n, where $0 \leq n \leq 56$. An integer D is announced, and players try to obtain two cards, which are not necessarily distinct, such that one of their differences (positive or negative) is congruent to D modulo 57. For example, if $D = 27$, then the pair 24 and 54 would work because $24 - 54 \equiv 27 \bmod 57$. Compute n such that this task is always possible for all D.

Answer: 43

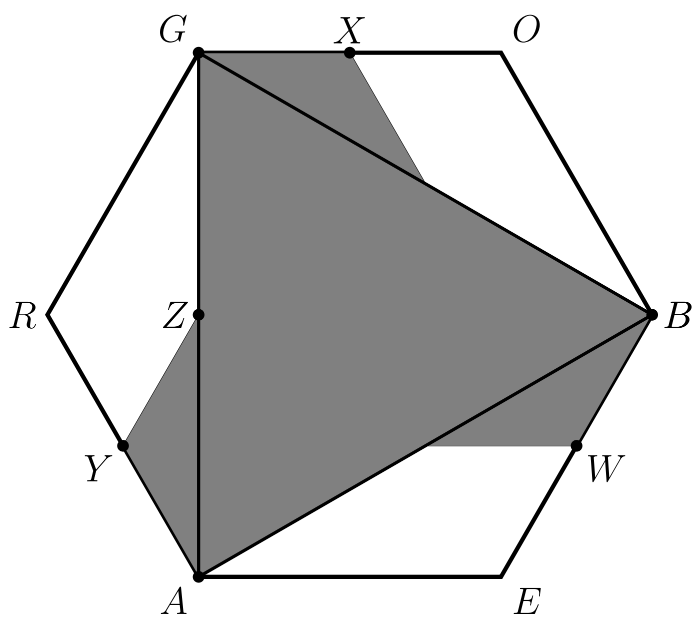
Solution: Denote the set of cards as the set $S = \{6, 11, 12, 14, 24, 47, 54, n\}$ Note that there are 56 ordered pairs (i, j), which must cover all 56 nonzero residues modulo 57, so each ordered pair must have a distinct difference modulo 57. Furthermore, since $i - j \equiv 57 - (j - i) \bmod 57$ and the function $f(x) = 57 - x$ is a 1-1 correspondence over the nonzero residues modulo 57, each unordered pair must have a distinct least difference from 1 to 28 (modulo 57). If we take the set $S - \{n\} = \{6, 11, 12, 14, 24, 47, 54\}$ and find what are not the least differences between these numbers, we can determine the least differences between these numbers and n.

Running through the first few possibilities, we see that while $1, 2, 3, 5, 6$, and 7 are the smallest differences that can be made from selecting two numbers in S and doing the task. We see 4 is not in this list so $n = a + 4$ or $n = a - 4$ for some $a \in S - \{n\}$ (this reads as the set S not containing the number n). We can now check the resulting 14 possibilities, quickly eliminating $6 \pm 4, 11 \pm 4, 12 \pm 4, 14 \pm 4, 24 - 4, 47 + 4$, and 54 ± 4 because they create duplicate differences that we've already seen. The remaining numbers to check are $24 + 4 = 28$ and $47 - 4 = 43$. We see 28 doesn't work since $28 - 11 \equiv 14 - 54 \pmod{57}$. We can exhaust all cases (for rigor) to see $n = \boxed{43}$.

21. On regular hexagon $GOBEAR$ with side length 2, bears are initially placed at G, B, A, forming an equilateral triangle. At time $t = 0$, all of them move clockwise along the sides of the hexagon at the same pace, stopping once they have each traveled 1 unit. What is the total area swept out by the triangle formed by the three bears during their journey?

Answer: $\frac{15\sqrt{3}}{4}$

Solution:



Let X, W, and Y be the midpoints of $\overline{GO}$, $\overline{BE}$, and $\overline{AR}$, respectively. The shaded area in the diagram represents the total area that is swept out as triangle $\triangle GBA$ rotates to triangle $\triangle XWY$. To calculate this area, we will instead subtract the three unshaded regions from the area of the entire hexagon. It suffices to just calculate one, as the other two are equivalent.

Let Z be the intersection of $\overline{GA}$ and $\overline{XY}$. As the two bears move from G to X and from A to Y, they form a line that continuously rotates and shifts from line $\overline{GA}$ to $\overline{XY}$. Thus, the region that is not swept out by this side is quadrilateral $GRYZ$. Note that since X, Y are the midpoints of $\overline{GO}, \overline{RA}$, respectively, we have $\overline{XY} \parallel \overline{GR}$. Thus, $AYZ \sim ARG$, with the ratio of similarity being $\frac{1}{2}$ since Y is the midpoint of $\overline{AR}$. Hence $[\triangle AYZ] = \frac{1}{4}[\triangle ARG]$. Finally, note that we can calculate $AZ = \sqrt{3}$ and $RZ = 1$ from the 30-60-90 triangle $\triangle ARZ$, so $[\triangle ARG] = AZ \cdot RZ = \sqrt{3}$.

Thus

$$[GRYZ] = [\triangle ARG] - [\triangle AYZ] = \frac{3}{4}[\triangle ARG] = \frac{3\sqrt{3}}{4}.$$

The area of the entire hexagon can be calculated as $\frac{3(2)^2\sqrt{3}}{2} = 6\sqrt{3}$, so the total area swept out by the three bears is $6\sqrt{3} - 3\left(\frac{3\sqrt{3}}{4}\right) = \boxed{\frac{15\sqrt{3}}{4}}$.

22. Given a positive integer n, let $s(n)$ denote the sum of the digits of n. Compute the largest positive integer n such that $n = s(n)^2 + 2s(n) - 2$.

Answer: 397

Solution: Let d denote the number of digits in n. Note that we cannot have $d \geq 5$: because $s(n) \leq 9d$, we must have

$$10^{d-1} \leq n \leq (9d)^2 + 2 \cdot 9d - 2.$$

In particular, $10^{5-1} > (9 \cdot 5)^2 + 2 \cdot 9 \cdot 5 - 2$, with the left-hand side increasing much faster than the right-hand side, so $d \geq 5$ do not satisfy the condition.

Additionally, we cannot have $d = 4$. Because $n \leq (9 \cdot 4)^2 + 2(9 \cdot 4) - 2 < 1400$, we have $s(n) \leq 1 + 3 + 9 + 9 = 22$, so

$$n \leq 22^2 + 2 \cdot 22 - 2 < 1000.$$

Therefore, n has at most 3 digits, so $s(n) \leq 9 \cdot 3 = 27$. Now, observe that

$$n = s(n)^2 + 2s(n) - 2 \equiv n^2 - n + 1 \pmod 3,$$

so $(n-1)^2 \equiv 0 \pmod 3$, and thus $n \equiv 1 \pmod 3$. We now do casework on $s(n)$.

- If $s(n) = 25$, then $n = 25^2 + 2 \cdot 25 - 2 = 673$, contradiction.
- If $s(n) = 22$, then $n = 22^2 + 2 \cdot 22 - 2 = 526$, contradiction.
- If $s(n) = 19$, then $n = 19^2 + 2 \cdot 19 - 2 = 397$, which works.
- If $s(n) < 19$, then $n < 397$, which is less.

Thus, the answer is $\boxed{397}$.

23. For real numbers B, M, and T, we have $B^2 + M^2 + T^2 = 2022$ and $B + M + T = 72$. Compute the sum of the minimum and maximum possible values of T.

Answer: 48

Solution: From the second equation, we have that $B + M = 72 - T$. Note that $BM \leq \frac{(B+M)^2}{4}$ by AM-GM (or by rearranging $(B-M)^2 \geq 0$), so:

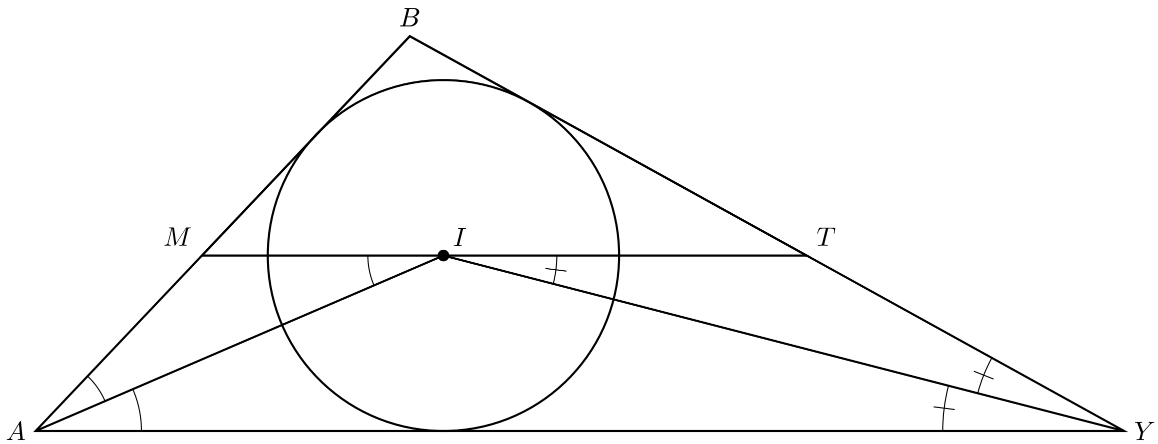
$$B^2 + M^2 = B^2 + M^2 + 2BM - 2BM \geq (B+M)^2 - \frac{(B+M)^2}{2} = \frac{(B+M)^2}{2} = \frac{(72-T)^2}{2}$$

Utilizing the first equation, we now have $\frac{(72-T)^2}{2} + T^2 \leq 2022$. Rearranging this inequality, we get $3T^2 - 144T + 5184 \leq 4044$. Dividing both sides by 3 and then factoring yields $(T-10)(T-38) \leq 0$, which implies that $10 \leq T \leq 38$. Thus, the minimum possible value of T is 10 and the maximum possible value of T is 38. Their sum is $\boxed{48}$. Indeed, we find that the triples $(B, M, T) = (17, 17, 38)$ and $(B, M, T) = (31, 31, 10)$ are solutions. As a remark, notice that T is minimized and maximized when $B = M$.

24. Triangle $\triangle BMT$ has $BM = 4$, $BT = 6$, and $MT = 8$. Point A lies on line $\overleftrightarrow{BM}$ and point Y lies on line $\overleftrightarrow{BT}$ such that $\overline{AY}$ is parallel to $\overline{MT}$ and the center of the circle inscribed in triangle $\triangle BAY$ lies on $\overline{MT}$. Compute AY.

Answer: $\mathbf{\frac{72}{5}}$ **or 14.4**

Solution:



Let I be the incenter of $\triangle BAY$. Since $\overline{AI}$ bisects $\angle YAM$ and $\overline{MI} \parallel \overline{AY}$, we have

$$\angle BAI = \angle YAI = \angle AIM,$$

so $AM = MI$. Likewise, $YT = TI$. By the Angle Bisector Theorem, $MI/IT = MB/BT = 2/3$, so $MI = 16/5$ and $IT = 24/5$. Since $\triangle BMT \sim \triangle BAY$,

$$AY = MT \cdot \frac{BA}{BM} = 8 \cdot \frac{16/5 + 4}{4} = \boxed{\frac{72}{5}}.$$

25. Bayus has eight slips of paper, which are labeled 1, 2, 4, 8, 16, 32, 64, and 128. Uniformly at random, he draws three slips with replacement; suppose the three slips he draws are labeled a, b, and c. What is the probability that Bayus can form a quadratic polynomial with coefficients a, b, and c, in some order, with 2 distinct real roots?

Answer: $\mathbf{\frac{111}{128}}$

Solution: We compute the complement: namely, we compute the probability that regardless of the ordering of a, b, and c, no quadratic Bayus makes will have 2 distinct real roots. For this to be the case, it is sufficient that the largest possible discriminant is nonpositive. Without loss of generality, assume $b = \max(a, b, c)$, so that the largest possible discriminant is $b^2 - 4ac$. Now, let $x = \log_2 a$, $y = \log_2 b$, and $z = \log_2 c$, so that x, y, z are integers satisfying $y = \max(x, y, z)$. Then

$$0 \geq b^2 - 4ac = 2^{2y} - 4 \cdot 2^x \cdot 2^z,$$

so

$$x + z + 2 \geq 2y \geq x + z.$$

Thus, $x + z \in \{2y, 2y - 1, 2y - 2\}$, so the unordered pair $\{x, z\}$ is one of $\{y, y\}$, $\{y, y - 1\}$, $\{y, y - 2\}$, or $\{y - 1, y - 1\}$.

Now, we lift our assumption that $y = \max(x, y, z)$ to compute the answer. We have four cases.

- Suppose (x, y, z) is some ordering of (t, t, t). There are 8 such ordered triples.
- Suppose (x, y, z) is some ordering of $(t, t, t-1)$. There are $3 \cdot 7 = 21$ such ordered triples.
- Suppose (x, y, z) is some ordering of $(t, t-1, t-1)$. There are $3 \cdot 7 = 21$ such ordered triples.
- Suppose (x, y, z) is some ordering of $(t, t, t-2)$. There are $3 \cdot 6 = 18$ such ordered triples.

Subtracting, the probability is $1 - \frac{8+21+21+18}{8^3} = \boxed{\frac{111}{128}}$.