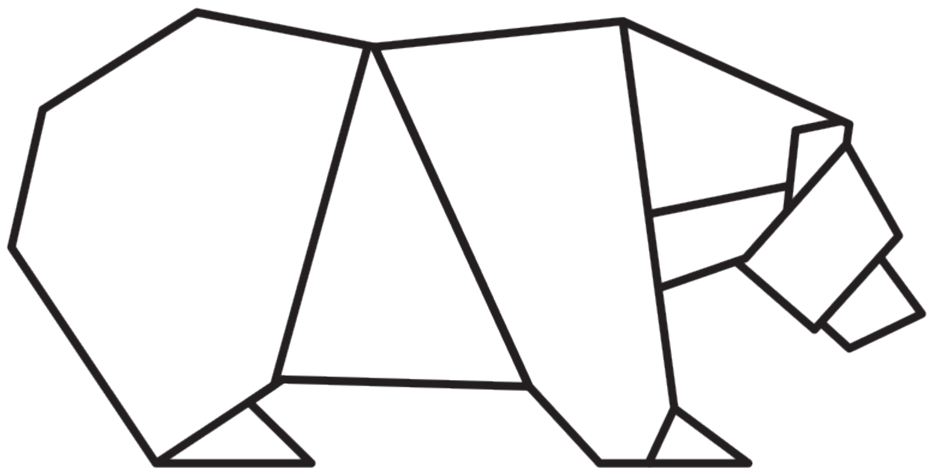
Berkeley Math Tournament 2024

Discrete Test



November 2, 2024

Time limit: 60 minutes.

Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written inside the boxes on the answer sheet will be considered for grading.

No calculators. Protractors, rulers, and compasses are permitted.

Answer format overview:

- Carry out any reasonable calculations. For instance, you should evaluate $\frac{1}{2}+\frac{1}{3}$, but you do not need to evaluate large powers such as 7^8.
- Write rational numbers in lowest terms. Decimals are also acceptable, provided they are exact. You may use constants such as π, e, $\sin(10^\circ)$, and $\ln 2$ in your answers.
- Move all square factors outside radicals. For example, write $3\sqrt{7}$ instead of $\sqrt{63}$.
- Denominators do *not* need to be rationalized. Both $\frac{\sqrt{2}}{2}$ and $\frac{1}{\sqrt{2}}$ are acceptable.
- Do not express an answer using a repeated sum or product.

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1. Find the third-largest three-digit multiple of three that is a palindrome. (Recall that a palindrome is a number that reads the same forward and backward, such as 444 or 838, but not 227.)

2. The 35-step staircase of Sather Tower is being renovated. Each step will be painted a single color such that the stairs repeat color every 5 steps. There are 14 available stair colors, including blue and gold. Both blue and gold must be used, and each color may only cover up to 10 steps. With these restrictions, in how many different ways can the stairs be colored?

3. Find the number of positive integers, n, such that $\dfrac{20+n}{24-n}$ is an integer.

4. Eight players are seated around a circular table. Each player is assigned to either Team Green or Team Yellow so that each team has at least one player. In how many ways can the players be assigned to the teams such that each player is on the same team as at least one player adjacent to them?

5. Clara has a pair of nine-sided fair dice, each of whose faces are labeled $1, 2, 3, \ldots, 9$. Justin also has a pair of nine-sided fair dice, and the faces of his dice have positive integer labels, but one of Justin's dice has the number 13. When Clara and Justin roll their dice, it turns out that for every number S, the probability that the sum of the results of Clara's dice is S is equal to the probability that the sum of the results of Justin's dice is S. What is the probability that when Justin's dice are rolled, the results of the dice are equal?

6. Find the greatest multiple of 43 whose base 6 representation is a permutation of the digits $1, 2, 3, 4,$ and 5. (Express your answer in base 10).

7. Gigi randomly rearranges four G's and seven I's to form an eleven-letter string. What is the probability that there is a group of four consecutive letters that form "$GIGI$," her name?

8. For an arbitrary positive integer n, we define $f(n)$ to be the number of ordered 5-tuples of positive integers, $(a_1, a_2, a_3, a_4, a_5)$, such that $a_1a_2a_3a_4a_5 \mid n$. Compute the sum of all n for which $f(n)/n$ is maximized.

9. Compute the remainder when
$$\left(4^{1^2} + 4^{2^2} + 4^{3^2} + \cdots + 4^{82^2}\right)^2$$
is divided by 83.

10. The positive integers 1 through 9 are placed in the 9 cells of a 3×3 grid. Then, for every pair of cells sharing a side, the sum of the numbers in that pair is recorded in a list. The most number of times any number occurs in the list is 4. In how many ways could numbers have been placed in the grid?