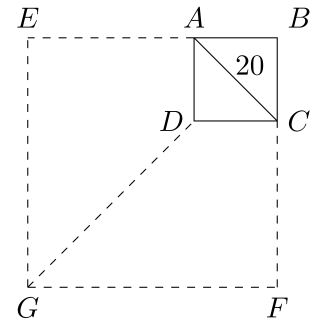
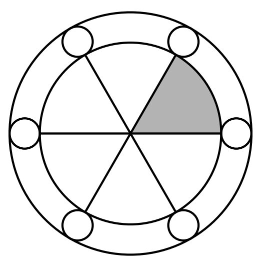
1. The area of square $EBFG$ is 9 times the area of square $ABCD$, as shown in the diagram below. The diagonal $\overline{AC}$ of $ABCD$ has length 20. Compute the length of $\overline{DG}$.



2. Wen writes a positive integer W on the board. Repeatedly, she multiplies this integer by 2, writes the result on the board, and erases the original number. At some point, the value written on the board is 2024. Compute the smallest possible value of W.

3. Compute the third largest factor of the third largest factor of the third smallest positive integer whose third largest factor has at least three factors. Recall that every positive integer is a factor of itself.

4. Jonathan is riding a unicycle. The unicycle wheel has a large outer circle, 6 small circles, and a medium inner circle divided into 6 congruent sectors by 3 spokes, as shown in the diagram below (diagram not to scale). The smallest circles have radius 1, the largest circle has radius 7. All six small circles are tangent to the inner circle at an endpoint of a spoke, and tangent to the outer circle. Compute the area of the shaded sector.

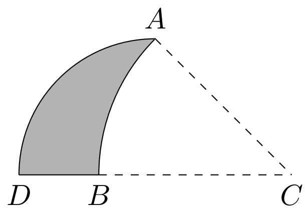


5. Compute the smallest three-digit positive integer with distinct nonzero digits satisfying the property that it is not divisible by any of its digits. For example, 426 does not have this property because it is divisible by 2 and 6.

6. Compute $(x+y)^{100}$ given

$$\left(\frac{x^2+2xy+y^2}{y+z}\right)^{100}=1, \quad \left(\frac{y^2+2yz+z^2}{x+z}\right)^{50}=8, \quad \left(\frac{x^2+2xz+z^2}{x+y}\right)^{25}=16.$$

7. During his escape from Alcatraz Island, Aditya swims to San Francisco and sees a shark fin above the water, indicated by the shaded area in the diagram below. The shark fin is formed by a quarter circle of radius 3 with arc $\widehat{AD}$ cut by its overlap with a $45°$ sector of the circle centered at C passing through A and B (on the two straight sides of the quarter circle). Compute the area of the shark fin.



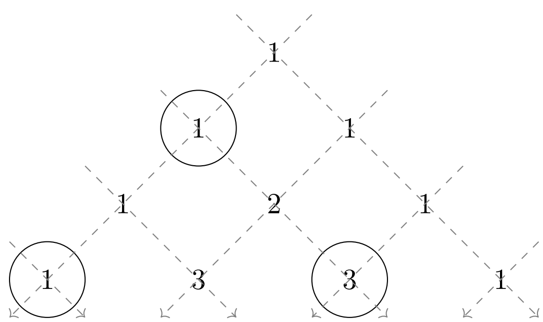
8. Isaac has a steel tube that is 2024 units long. He wants to cut this tube into C smaller pieces such that no three pieces can be the sides of a triangle with positive area, and each piece has a unique positive integer length. Compute the greatest possible value of C.

9. Let $\log_2^*(n)$ be the number of times we need to apply $\log_2$ to n to get a number less than 1. This function grows very slowly, but a useful application is to extended exponentiation:
Let $a \uparrow^1 b = a^b$, and $a \uparrow^{n+1} b = \underbrace{a \uparrow^n (a \uparrow^n (\cdots (a \uparrow^n a)))}_{b-1 \ \ \uparrow^n\text{'s}}$. For instance, $4 \uparrow^2 3 = 4 \uparrow^1 (4 \uparrow^1 4)$.
Compute $\log_2^*(2 \uparrow^3 4)$.

10. Suppose N, a, and b are positive integers such that $N = a^3 + b^3 - a^2b - ab^2$. Given that N has exactly 6 factors, compute the least possible value of N.

11. Let $\triangle ABC$ be a right triangle such that $\angle B = 90°$. Points D and E are placed on $\overline{AC}$ such that $AB = AE$ and $BC = DC$. Given that $AD = 2$ and $EC = 9$, compute $BD \cdot BE$.

12. Define an *upright* triangle to be a set of three distinct vertices on Pascal's Triangle, where two vertices are on the same row and the third vertex is above both points and shares a diagonal with each of them. For example, the diagram below shows the first four rows of Pascal's triangle, and the three circled numbers are vertices of an upright triangle. Compute the sum of the vertices over all upright triangles on the first 10 rows of Pascal's Triangle.



13. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Additionally, let $\{x\}$ denote $x - \lfloor x \rfloor$. For example, $\lfloor \pi \rfloor = 3$ and $\{\pi\} = 0.1415\ldots$. Compute the integer n such that there are exactly 2024 positive solutions x to the equation

$$x^{\lfloor x \rfloor^{\{x\}}} = n.$$

14. For each positive integer n, let $s(n)$ be the sum of its digits and let $p(n)$ be the product of its digits. Compute the number of positive integers $n \leq 10^6$ that satisfy $s(n) - p(n) = 5$.

15. Arjun considers the parabola described by $y = x^2 - 2x + 2$ in the coordinate plane and chooses some θ uniformly at random from the interval $[0, 2\pi)$. He then rotates the parabola about its vertex counterclockwise by θ and counts the number of times the resulting parabola intersects the coordinate axes. The probability that there are exactly four intersections can be expressed as $\frac{\pi+\arcsin(r)-\arccos(r)}{2\pi}$ for some real number r. Compute r.

16. Let H be the orthocenter of obtuse triangle $\triangle ABC$ and D be the midpoint of $\overline{AC}$. A line through H perpendicular to $\overleftrightarrow{DH}$ intersects with $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ at points E and F, respectively. Given that $EH = 3$ and $HD = 5$, compute the maximum possible area of triangle $\triangle DEF$.

17. Let the *power*, $p(n)$, of a positive integer n be the number of fractions of the form $\frac{m}{n}$ that are in simplest form over all positive integers m with $1 \leq m \leq n$. ($\frac{1}{1}$ is in simplest form.) Let a positive integer n be *weak* if $\frac{p(n)}{n} \leq \frac{p(k)}{k}$ for all $1 \leq k < n$. Compute the sum of all *weak* positive integers less than 2024.

18. Arthur has a four-sided die, of which all faces are initially labeled 1. Every second, Arthur rolls the die and, if the outcome of the die is 1, then he changes the number on the top face to 2, and otherwise, he changes the number to 1. Let the probability that the sum of the die rolls is at some point k be p_k. Let $\lfloor x \rfloor$ denote the largest integer less than or equal to x, and let $\{x\} = x - \lfloor x \rfloor$. Compute $\{p_1 + p_2 + \cdots + p_{2024}\}$.

19. Let N_{21} be the answer to problem 21. Let b and d be real numbers such that the polynomial $P(x) = x^4 - N_{21}x^3 + bx^2 - x + d$ has real roots p, q, r, s and $pq = rs$. Compute the greatest possible integer value of $P(N_{21})$.

20. Let N_{19} be the answer to problem 19. Danielle picks a real number p uniformly at random from $[0, 1]$. She then creates a magic coin that has probability p of landing on heads and probability $1-p$ of landing on tails when flipped. Compute the probability that Danielle lands heads exactly N_{19} times in $5N_{19}$ flips of the coin.

21. Let N_{20} be the answer to problem 20. In triangle $\triangle ABC$, the angle bisector of $\angle B$ intersects $\overline{AC}$ at D. The perpendicular bisector of $\overline{BD}$ intersects $\overline{AB}$ and $\overline{BC}$ at X and Y respectively. The area of $\triangle BXY$ is $\frac{7}{55}$, $AD = \frac{1}{4}$, and $DC = N_{20}$. Compute the area of $\triangle ABC$.

22. Tetrahedron $ABIH$ has points C on $\overline{BH}$, D on $\overline{AH}$, E on $\overline{HI}$, F on $\overline{AI}$, and G on $\overline{BI}$ such that the squares $BCEG$ and $ADEF$ have side length 5. Quadrilaterals $ABCD$ and $ABGF$ are isosceles trapezoids. Given that $AB = 4\sqrt{2}$ and $CD = 3\sqrt{2}$, compute the volume of solid $ABCDEFG$.

23. Let X denote the set $\{-1, 0, 1, 2, 3, 4\}$, and let $\mathcal{P}(X)$ denote the set of all subsets of X. Compute the number of functions $f\colon \mathcal{P}(X) \to X$ such that $f(\varnothing) = 0$ and $f(A \cap B) + f(A \cup B) = f(A) + f(B)$ for any subsets A and B of X.

24. A polynomial with integer coefficients that has a root of the form $k\cos\left(\frac{4\pi}{7}\right)$ for some positive integer k is called *simple* if there are no polynomials of lesser degree with integer coefficents sharing the same root. There exists a unique simple polynomial $P(x)$ with leading coefficient 1 such that $|P(3)|$ is minimized over all simple polynomials with leading coefficient 1. Compute $P(4)$.

25. The Fibonacci numbers F_n for integers $n \geq 1$ are defined as follows: $F_1 = F_2 = 1$, and for $n > 2$, $F_n = F_{n-1} + F_{n-2}$. Kiran makes a list of all the distinct positive integers less than or equal to

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10^6 that can be expressed as the sum of at most four distinct Fibonacci numbers. Compute the length of Kiran's list. Submit your answer as an integer E; if the correct answer is A, your score for this question will be $\max\left(0, 25 - \left\lfloor \frac{\sqrt{|A-E|}}{4} \right\rfloor\right)$.

26. Submit a positive real number c to at most 6 decimal places. Define the function $f^1(x) = x^2 + c$, and let $f^k(0) = f^1(f^{k-1}(0))$ for $k \geq 2$. Let N be the smallest positive integer such that $f^N(0) > 2024$. If such an N does not exist, your score is 0 points. Otherwise, your score is $\max(0, 25 - 3|N - 20|)$ points.

27. Oliver rolls a standard, fair 6-sided die 640 times. Oliver tells Tushar that no 5s were rolled before he rolled a 6 (it is possible he rolled no 5s or 6s). Given this information, Tushar computes x, the expected number of 6s rolled by Oliver. Compute $\log_5(\lceil 6x \rceil - 6x)$. Submit your answer as a real number E to at most 3 decimal places; if the correct answer is A, you will receive $\max(0, 25 - \lfloor 4|A - E| \rfloor)$ points.