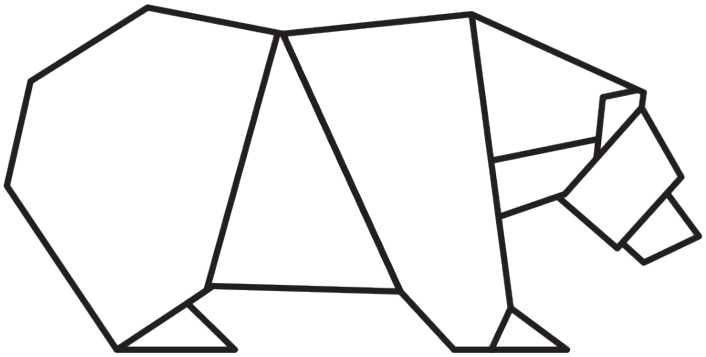
Berkeley Math Tournament 2024

Power Round



November 2, 2024

Time limit: 90 minutes.

Maximum score: 185 points.

Instructions: For this test, you work in teams to solve multi-part, proof-oriented questions. Problems that use the words “compute,” “find,” “draw,” or “write” require only an answer; no explanation or proof is needed. **Problems that use the words “construct” or “fold” take origami submissions, as described in the Preliminaries.** Unless otherwise stated, all other questions require explanation or proof. **Please make sure to read the Preliminaries section on the back of this cover page after time has begun for further important submission instructions.**

Answers should be written on sheets of blank paper, clearly labeled, in order, with problem numbers in the top right corners. If you have multiple pages for a problem, number them and write the total number of pages for the problem (e.g. 1/2, 2/2).

Write your team ID number clearly on each sheet. Only submit one set of solutions for the team. Do not turn in any scratch work. After the test, put the sheets you want graded into your team envelope in order by problem number.

The problems are ordered by content, *not difficulty*. The difficulties of the problems are generally indicated by the point values assigned to them; it is to your advantage to attempt problems throughout the test. In your solution for a given problem, you may cite the statements of earlier problems (but not later ones) without additional justification, even if you haven’t solved them.

No calculators. Protractors, rulers, and compasses are permitted.

DO NOT DISCUSS OR DISTRIBUTE ONLINE UNTIL DECEMBER 8, 2024

0 Preliminaries

The theme of this power round is origami! Before beginning, we introduce some rules and guidelines regarding submissions to certain problems.

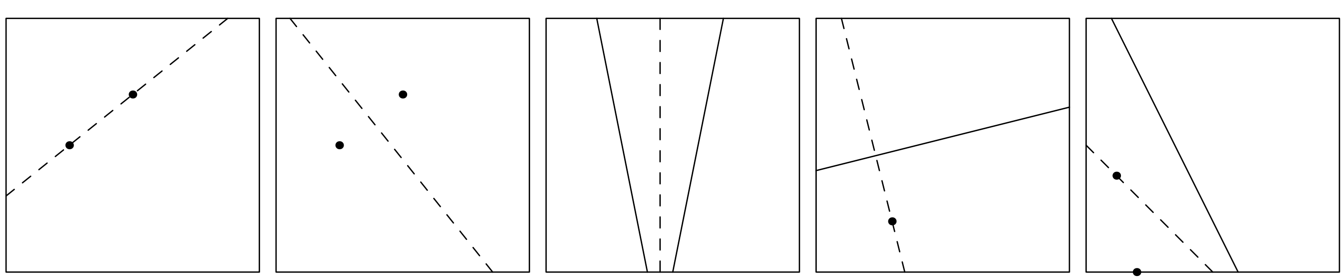
Recall that, as stated in the instructions, problems that use the words "construct" or "fold" take origami submissions. To submit your answers for any origami constructions, you may do either of the following:

- Submit a **folded** piece of origami, with all intentional creases also **clearly marked and labeled with pen or pencil**. See the crease pattern at the bottom of page 4 to see what the origami should look like when unfolded. The problem number must be clearly written somewhere on the piece. In addition, number every fold in the order of folding.
- List a sequence of folds with diagrams as needed on your submission paper. Each fold should be labeled (A) through (E) as described below, and the ordering of the folds should be clearly labeled. For example: "Step 1: Fold the top-left corner to the bottom-right corner using fold (B)." If a fold creates multiple crease lines, label each crease line with the same number in the sequence.

The rules for folding submittable origami are as follows:

- **A fold on the paper will create one or more *crease lines*, which are lines created by paper being folded onto itself. These lines, as well as their intersections with other lines, may be used as reference points in later steps.**
- **You can apply folds to an already-folded paper, but you may only make one crease at a time.**
- **Unfolding along existing crease lines does not count as a fold.**
- The following rules describe all "allowable" folds for origami submissions in Chapters 1 and 2:
  - **(A)** If you have two known points, you may create the fold passing through both points.
  - **(B)** If you have two known points, you may create the crease that folds one point onto the other.
  - **(C)** If you have two known lines, you may fold one line onto the other.
  - **(D)** If you have a point and a line, you may create the fold perpendicular to the line, passing through the point.
  - **(E)** If you have two points and a line, you may fold one of the points to the line via a crease that passes through the other point.
- In particular, you may not:
  - Fold a point arbitrarily onto a line
  - Fold a line arbitrarily onto a point
  - Make two folds simultaneously (e.g. folding a paper into thirds with one "fold")

Example Folds:



(A) (B) (C) (D) (E)

1 Constructions (40 pts)

Before beginning the round, please make sure you have carefully read the Preliminaries on the back of the cover sheet.

Welcome to the power round! Let's take some time to explore something you might have become familiar with years ago: origami. Origami is a source of several interesting problems in mathematics, with deep connections to topics like graph theory, solving polynomials, and field theory.

In this section, let's think about how origami can be used to build upon the classic constructions of a straightedge and compass.

Question 1.1. (3 pts) In the fewest number of folds possible, fold a paper so that when it is unfolded, the crease lines divide the paper into eight congruent triangles. Additionally, explain why this is the fewest number of folds possible. (Hint: It's not 4.)

Constructions can be very difficult, so it is often useful to first consider the geometry of the problem. Ask yourself what shapes or segments would be useful to have in order to make the construction. Let's break down our first construction into a few different parts. In the next few questions, we will be constructing a square with $3/4$ the area of our original square.

Question 1.2. (2 pts) Given a square with side length l and area A, find the side length x of a square with area $\frac{3}{4}A$ in terms of l.

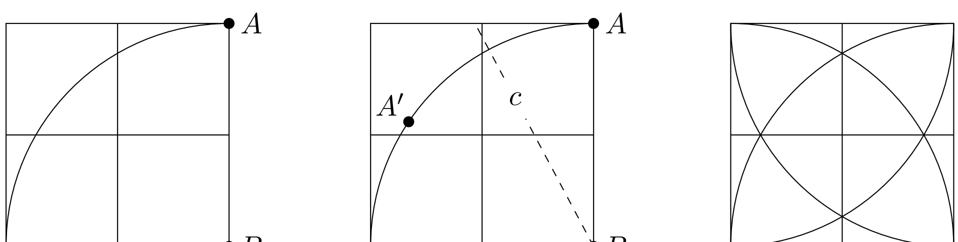
The next thing we want to think about is how we could construct the side length we want. While there are many different shapes that could be folded, we focus on the simplest ones.

Question 1.3. (2 pts) Show there exists a right triangle with side lengths $x, l, l/2$.

Our goal now is to make folds that correctly measure the above lengths at the correct angle. Let's consider the geometry of the square and make some conclusions about where the triangle must be located. Importantly, where must a vertex of the side with length l be, and how can you measure length $l/2$?

The only way to create a length l ending at a vertex B of the square is to rotate a side length $\overline{AB}$ about the vertex B. To do this, we can reflect the vertex A across a line c through B, creating A'. Conveniently, this is the same thing as creasing along c! The set of all points length l from a vertex is just the circle with center B, and radius l. Considering only the parts of these circles intersecting the paper, we note that all points on the following arcs are (theoretically) constructible, but we don't have enough information to accurately fold all of them. Now, we want to create $l/2$. The only length we have now is l, so we need to divide this in half. We can do this by folding horizontally and vertically in half. These lines will be distance $l/2$ from the parallel sides of the square.

As a corollary, if A rotates by an angle θ, the crease line will form an angle $\theta/2$ with the original line $\overline{AB}$.



(a) (b) (c)

Question 1.4. (2 pts) Draw a possible orientation and location of the right triangle found in Question 1.3 with vertices on the lines and curves given in diagram (c).

We now have a possible diagram for our construction of a segment with length x. Now it's time to determine whether this diagram is foldable.

Question 1.5. (5 pts) Given a square piece of paper, construct a square with exactly $3/4$ the area of the original in at most eight folds.

Origami can be used to create other interesting polygons as well.

Question 1.6. (5 pts) Given a square piece of paper, draw an equilateral triangle with the greatest possible area and prove that its area is indeed maximal. Additionally, fold this triangle.

Notice that in these cases, the main challenge was constructing a segment of a certain length. This is a way of thinking about origami that may be new, so let's try to make some other segments as well. Importantly, we want to try to make segments of arbitrary rational length a/b. These measurements are quite useful to people actually making origami, so it's good to have a way to construct them.

The main difficulty is constructing a segment of length $1/n$, where n is odd. For $a > 1$, we can just take a lengths of $1/n$, and dividing a length in half is easy (we can just fold in half), so a length $\frac{1}{2k}$ for some k is trivial after $1/k$ is created.

We begin our construction of segments with an example that may appear trivial: dividing a paper into equal thirds. However, the "classic" method of lining up both folds at once and pressing the paper flat violates one of our rules: that only one fold may be made at a time.

Question 1.7. (6 pts) Fold a piece of paper into thirds in five folds, making exactly one fold at a time. *(Hint: create two similar triangles with side length ratio 1:2)*

Before diving into the main problem of a/b, let's work on a more specific case. Some side lengths are easy to create (there are ways to create 1/3, 1/5, 1/7, for instance), so let's try to use these to create some more segments. Specifically, we want to create a segment of length $\frac{1}{2k-1}$ given a segment of length $\frac{1}{2k}$.

Note that $[0,1] \times [0,1] \subset \mathbb{R}^2$ is the coordinate plane with x- and y-values between 0 and 1 inclusive.

Question 1.8. (3 pts) Given a square sheet of origami paper defined as $[0,1] \times [0,1]$, let $A = \left(0, \frac{1}{2k}\right)$ and $B = \left(\frac{1}{2k}, 1\right)$. Now, fold A onto B, with the crease line L intersecting the y-axis at point $P = (0, P)$. Prove that the distance from P to $(0,1)$ is $\frac{k-1}{2k-1}$.

That was certainly not obvious to begin with, but with this fold we can now generalize to a length $\frac{a}{2k-1}$.

Question 1.9. (4 pts) Given an algorithm for dividing a segment into k segments of equal length, give an algorithm for constructing a length $\frac{a}{2k-1}$ for any integer a between 1 and $2k-1$.

This algorithm, while weaker than a general rational length generator, will give us insight into how we can create a more general algorithm.

Question 1.10. **(8 pts)** Given a square sheet of origami paper defined as $[0, 1] \times [0, 1] \subset \mathbb{R}^2$, find, with proof, an algorithm to create the point $(0, a/b)$.
Hint: A length we can always create is $b/2^{k+1}$, where 2^k is the largest power of 2 smaller than b.

2 Flat Folding (53 pts)

Now that we've spent some time considering constructible segments and shapes using origami, let's dive into the theory behind origami, and figure out when different origami patterns have certain properties. The main focus of this section will be considering when origami can fold flat. Determining whether a random assortment of creases folds flat is an extremely difficult task (NP-hard, in fact), but we can take a look at some specific cases.

Before we can start, let's go over some quick graph theory and origami definitions.

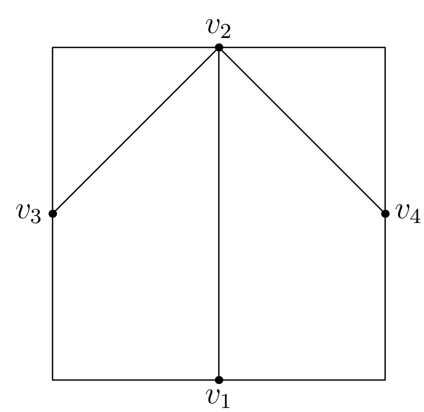
Definition 2.1. A **graph** $G = (V, E)$ is a set of vertices, V, and a set of edges, E. Each edge is itself an (unordered) set of two distinct vertices. Vertices that share an edge between them are called **adjacent**. The number of vertices in a graph is denoted $|V|$ and the number of edges is denoted $|E|$. The **degree** of a vertex is the number of edges connected to the vertex.

Definition 2.2. Given a piece of paper $R \subset \mathbb{R}^2$, a **crease pattern on** R is a graph $G = \{V, E\}$ with vertices V and creases E. Vertices on the boundary of R are called **boundary vertices** and vertices in the interior of R are called **interior vertices**. The **faces** of the crease pattern G are polygons separated by the crease lines E. Note that only final creases must be in the crease pattern– if an intermediate crease is used but ends up not folded, it is not part of the crease pattern.

Definition 2.3. A **mountain fold** is a crease on a piece of paper which, when viewed from directly above the crease, makes the crease lie above the faces on either side. Intuitively, it is a fold that creates a "mountain" shape.
A **valley fold** is a crease on a piece of paper which, when viewed from directly above the crease, makes the crease lie below the faces on either side.
A valley fold is the opposite of a mountain fold, in that a mountain fold will look like a valley fold when viewed from below and vice versa.

Note that the paper is defined as a subset of $\mathbb{R}^2$, but it is not necessary to define each point as a coordinate pair when simply drawing a crease pattern. Below is the crease pattern after the first step of a simple paper airplane on paper $[0, 1] \times [0, 1]$.

$$V = \left\{ \left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, 1\right), \left(0, \frac{1}{2}\right), \left(1, \frac{1}{2}\right) \right\}$$
$$E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}\}$$



Question 2.1. (2 pts) Given a square paper with the bottom left corner at $(0, 0)$ and the top right corner at $(1, 1)$, give a crease pattern G with exactly one interior vertex and four edges, and draw the crease pattern.

Now, let's introduce the main topic of this section: folding origami flat. Intuitively, some crease patterns will be able to be folded flat and some will not be. How can we determine what is possible and what isn't?

Definition 2.4. A crease pattern $G = (V, E)$ **folds flat** if, after folding along all crease lines and nowhere else (creating no new creases), the resulting origami would be a two-dimensional polygon if the paper had zero thickness.

Now that we know how to write crease patterns and what it means for something to fold flat, let's think about how we could formalize the idea of folding along these lines to create origami. The two most basic ways to fold paper are *mountain folds* and *valley folds*, and we will define a way to assign these folds to our crease patterns now.

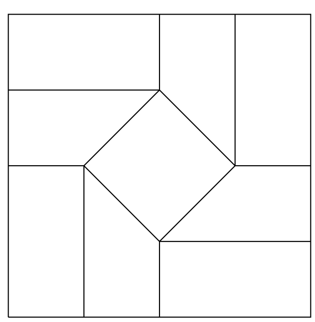
Definition 2.5. A **mountain-valley assignment** (or **MV-assignment**) for a crease pattern $G = (V, E)$ is a function $\mu : E \to \{-1, 1\}$ that assigns each crease line c in E a folding angle of $\mu(c)\pi$ from the horizontal. An MV-assignment is *valid* if the crease pattern can be folded flat.

Intuitively, a crease is a mountain fold if it is assigned -1 under μ and a valley fold if it is assigned 1.

For our purposes, it is sufficient to simply label each crease with 1 or -1 when given a crease pattern.

Question 2.2. (2 pts) Does a valid MV-assignment exist for the paper airplane crease pattern given after Definition 2.1? If so, give a valid MV-assignment. If not, explain why.

Question 2.3. (4 pts) Does a valid MV-assignment exist for the **square twist** crease pattern shown below? If so, give a valid MV-assignment. If not, explain why. *(Hint: Actually try to fold the shape!)*



Question 2.4. (3 pts) Explain why the faces of a flat-foldable origami crease pattern must be 2-colorable: that is, each face can be assigned one of two colors in a way such that no two adjacent faces will have the same color. A rigorous proof is not necessary.
The reverse (that a 2-colorable origami crease pattern is flat-foldable) is not necessarily true.

Before we continue, let's note some potentially useful proof techniques while working with origami. Proving theorems about origami will be very different than other subjects you may be familiar with. It may also valuable to consider "tracing" the cross-section of folded origami. Drawing a circle of radius 1 around a flat vertex fold (scaling if needed) and analyzing the image of this circle after folding the vertex flat may also give insights into the shapes. What directions does the curve go? What distance must it travel?

Now, let's turn our attention inwards and consider interior vertices. Flat-foldability is a very interesting property, so it stands to reason that we can make some observations about the interior vertices of a shape that folds flat. Let's narrow our scope a bit and think about single vertices.

Definition 2.6. A **single-vertex fold** is a crease pattern with exactly one interior vertex. A **flat vertex fold** is a single vertex fold that folds flat.

Notice that any crease pattern could be made up of many single-vertex folds.

Question 2.5. (3 pts) Prove that the degree (number of edges) of the interior vertex in a flat vertex fold must be even. (You may not use any results or theorems stated later in this section.)

We now know a lot about valid MV-assignments and have some criteria for flat-folding shapes. Let's use this to think about how mountain and valley folds must interact to fold flat. Here's a few tools that may prove useful.

Definition 2.7. Given a flat vertex fold $G = (V, E)$, let $E = \{l_0, l_1, ..., l_{2n-1}\}$ be the creases meeting at the interior vertex in clockwise order and let α_i be the angle between the creases l_i and l_{i+1} (α_0 is in between l_0 and l_{2n-1}). The **angle sequence** (α_i) of G is the sequence $(\alpha_0, ..., \alpha_{2n-1})$.

Lemma 2.8. Let $(\alpha_0, \alpha_1, ..., \alpha_{2n-1})$ be a sequence of $2n$ positive real numbers satisfying $\alpha_0 - \alpha_1 + \alpha_2 - \cdots - \alpha_{2n-1} = 0$. Then there exists an integer $0 \leq k \leq 2n - i$ such that $\alpha_k - \alpha_{k+1} + \alpha_{k+2} - \cdots \pm \alpha_i \geq 0$ for all $k < i \leq 2n - 1$ and $\alpha_k + \alpha_{k-1} - \alpha_{k+2} + \cdots \pm \alpha_i \geq 0$ for all $0 \leq i < k$.

Theorem 2.9. Kawasaki's Theorem: A single vertex crease pattern with angle sequence (α_i) folds flat if and only if

$$\sum_{i=0}^{2n-1} (-1)^i \alpha_i = 0.$$

That is, if and only if the alternating sum of angles is zero.

Lemma 2.10. (Big-little-Big Lemma) Let G be a flat vertex fold with angle sequence (α_i) and a valid MV-assignment μ. If, for some i, we have $\alpha_{i-1} > \alpha_i < \alpha_{i+1}$, then $\mu(l_i) \neq \mu(l_{i+1})$.

Theorem 2.11. Maekawa's Theorem: The difference between the number of mountain and valley folds in a **flat vertex fold** is 2.

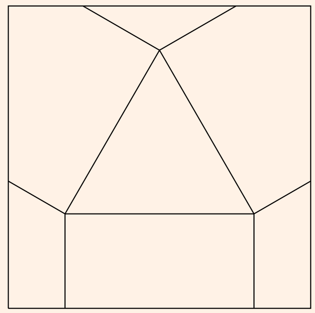
Question 2.6. (12 pts) Prove Kawasaki's Theorem.

Question 2.7. (3 pts) Prove that the sum of every other angle in a flat vertex fold is 180°.

Question 2.8. (8 pts) Prove Maekawa's Theorem.

Recall our main goal: how can we determine what folds flat and what doesn't?

Question 2.9. (4 pts) Is the following crease pattern flat-foldable? If so, provide a valid MV-assignment. If not, explain why. This diagram is to scale. (You may not use any later results in the section.)



We now will attempt to answer the question of flat-foldability for some specific crease patterns we may come across.

Definition 2.12. A crease pattern G is a **phantom fold** if all interior vertices satisfy the criterion described in Theorem 2.9 (Kawasaki's Theorem).

That is, a crease pattern is a phantom fold if each individual vertex could fold flat. It may or may not be globally flat-foldable, but this is definitely a step in the right direction when determining flat-foldability.

Let's now return to graphs.

Definition 2.13. Given a phantom fold $G = (V, E)$, the **origami line graph** $G_L = (V_L, E_L)$ is created as follows:
Let our initial set of vertices V_L be the midpoints of the creases $\{c_1, ..., c_n\}$ in G. Then,

- For each pair of creases $c_i, c_j \in E$, if they are forced to have different MV parity, let $\{c_i, c_j\} \in E_L$. (that is, connect the vertices associated with the two creases to each other).
- For each pair of creases $c_i, c_j \in E$, if they are forced to have the same MV parity and are *not* already the ends of a path of even length from performing the first step, add a new vertex $v_{i,j}$ to V_L and let $\{c_i, v_{i,j}\}, \{v_{i,j}, c_j\} \in E_L$.

Question 2.10. (4 pts) Verify that the crease pattern in Question 2.9 is a phantom fold and draw its origami line graph.

Finally, we are ready to make two important discoveries about flat-foldability!

Question 2.11. (3 pts) Prove that if the origami line graph G_L of a phantom fold G is not 2-vertex colorable, then G is not flat-foldable. A graph is 2-vertex colorable if each vertex can be assigned one of two colors in a way such that no two adjacent vertices will have the same color.

For some crease patterns, the origami line graph G_L completely determines MV-assignments. Because of this, we are encouraged to use these line graphs to make claims about the number of valid MV-assignments.

Question 2.12. (5 pts) Let C be a flat-foldable crease pattern with valid MV-assignments completely determined by C_L, and let n be the number of connected components of C_L. Find (with proof) the number of valid MV-assignments of C.

Recall that each valid MV-assignment is a unique way of folding a crease pattern. With this idea, we're now capable of figuring out how we can fold (some) crease patterns just by taking a good look at them!

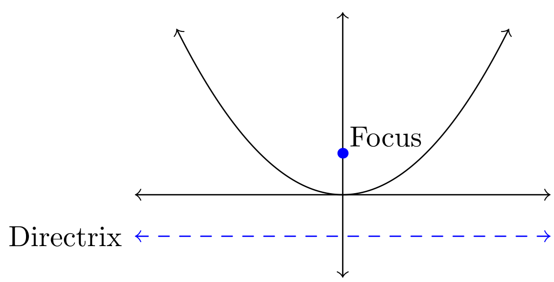
3 Solving Equations Using Origami (44 pts)

As it turns out, folding papers in certain ways can be used to solve polynomial equations.

Before getting into that, we'll first need to properly define a parabola. You may have seen that a parabola is a specific curve on a graph given by an equation like $y = x^2$.

In order to leverage origami theory to solve difficult equations, we'll need to use the rigorous geometric definition of a parabola, using the focus and the directrix.

Definition 3.1. A **parabola** is the set of all points in a plane that are equidistant from a fixed point, called the **focus**, and a fixed line, called the **directrix**.



In the case of the graph $y = x^2$, we see that our focus is $\left(0, \frac{1}{4}\right)$, and our directrix is $y = -\frac{1}{4}$. It turns out there's a pretty slick way to finding the focus and directrix for any parabola.

Theorem 3.2. For a quadratic that can be written in the form

$$(x - h)^2 = 4p(y - k)$$

for real numbers h, k and p, the focus of the corresponding parabola is $(h, k + p)$, and the directrix is $y = k - p$.

Question 3.1. (2 pts) Compute the focus and the directrix of each of the following two parabolas:

$$y = x^2 + 5, \tag{1}$$

$$16y - 3x^2 = 32. \tag{2}$$

Now that we've defined these terms, we can start diving deep into how origami theory fits in all of this:

Theorem 3.3. Folding a point P to a line L and then unfolding will create a crease line tangent to the parabola with focus P and directrix L.

By applying Theorem 3.3 again and again an infinite number of times, choosing different points on the left and right edges of the paper such that we make a crease at this point which will fold P onto L will eventually trace out a parabola on our paper with focus P and directrix L.

The proof for Theorem 3.3 uses a bit of geometry, and will be split between the following two questions. Assume we name the parabola E.

Question 3.2. (3 pts) Consider our fold that places P on line L. Let P' be the point on L to which P is folded to, and call the crease line C. Then, let X be the point on C such that $\overline{XP'}$ is perpendicular to L. Show that X is a point on our parabola E.

Question 3.3. (3 pts) Next, show that this point X is the only point on both our crease line C and our parabola E, thus showing that C is tangent to E.

Now that we have this theorem in our toolkit, we can observe a corresponding, tangible application by attempting to find the real roots of $f(x) = x^2 + ax + b$, where a and b are rational numbers.

Question 3.4. (2 pts) Compute the focus and directrix of the parabola $y = x^2 + ax + b$.

Call the focus for the above parabola P and the directrix L, where L is of the form $y = k$ for some constant k. Using similar logic from above, we can fold P to some arbitrary point (t, k) on L. This ends up creating a crease line with the following equation, derived through calculus:

$$y = (2t + a)x - t^2 + b.$$

The values of t that allow our crease line to be tangent to the parabola at a root are, assuming both roots are real, $t = \frac{-a+\sqrt{a^2-4b}}{2}$ and $t = \frac{-a-\sqrt{a^2-4b}}{2}$.

Substituting one of those values of t into our crease equation results in the following messy equation:

$$y = x\sqrt{a^2 - 4b} + \frac{a}{2}\sqrt{a^2 - 4b} + 2b - \frac{a^2}{2}.$$

Question 3.5. (5 pts) Our final step is to find a point P' on our crease line that is easy to construct from the coefficients of our original quadratic. Find this point P', and then explain how the root can be found using P' (and other previous information). Assume that the x-axis is an already-constructed line. *Hint: what value of x would drastically simplify the above line equation?*

Here's a less algebra-heavy way to solve for the real roots of $x^2 + ax + b = 0$, where a and b are rationals, known as **Lill's construction**:

1. On a graph, construct $A = (0, 1)$, and $B = (-a, b)$.
2. Draw a circle with diameter $\overline{AB}$ centered at the midpoint, C, of $\overline{AB}$.
3. If M and N are the two points where this circle intersects the x-axis, then assuming our origin is O, the x-coordinates of M and N, should they exist, will be solutions to $x^2 + ax + b = 0$.

Question 3.6. (3 pts) Show how the points C, M, and N can be constructed through folding. Assume that the y-axis and x-axis have been constructed already, and that M and N are both real points.

Question 3.7. (4 pts) Show that the x-coordinates of M and N correspond to the roots of the quadratic equation $x^2 + ax + b = 0$.

We have seen that origami, like straightedge and compass constructions, can solve quadratic equations. However, origami is more powerful than that, as it can also construct arbitrary cube roots. One way to do this is through the Beloch square.

Definition 3.4. Given two points, A and B, and two lines, r and s, the **Beloch square** is the square $WXYZ$ such that the two adjacent corners X and Y lie on r and s, respectively, and the sides $\overline{WX}$ and $\overline{YZ}$, or their extensions, pass through A and B, respectively.

Before we get into the Beloch square, however, we can first dive into the Beloch fold.

Definition 3.5. Given two points, A and B, and two lines, r and s, the **Beloch fold** is the single fold that places A onto r and B onto s simultaneously.

Question 3.8. (4 pts) Make a connection between the Beloch fold and Theorem 3.3. In terms of parabolas, what is the Beloch fold really doing?

Question 3.9. (8 pts) Given two points A and B and two lines r and s, detail a series of folding steps for constructing a Beloch square $WXYZ$ as detailed in Definition 3.4. Note: do not submit folded origami.

Question 3.10. (10 pts) Take r to be the y-axis, and take s to be the x-axis. Then, let $A = (-1, 0)$ and $B = (0, -2)$. If r' and s' are constructed to be the lines $x = 1$ and $y = 2$, detail a series of folds that end up allowing us to construct the cube root of two within this setup.

In addition to constructing cube roots, Beloch's square also gives way to constructing solutions to cubic equations. Check out Lill's Method for further reading on this idea!

Theorem 3.6. For real numbers a, b, and c, if r is a real solution to $x^3 + ax^2 + bx + c = 0$, then given $(0, a)$, $(0, b)$, and $(0, c)$, it is possible to construct $(0, r)$ by folding.

4 The Bounds of Foldability (48 pts)

In this last chapter, we'll be exploring the bounds of what is and isn't foldable. Before we get started on looking into what types of shapes we're able to fold, we first look into what types of points we can construct, also known as the **origami numbers**.

To do this, we'll be visualizing an **infinitely large** sheet of paper, in all directions. After drawing a horizontal and vertical axis, we mark the points $(0, 0)$ and $(1, 0)$ on this sheet of paper. When we fold and unfold this paper (using straightedges), it will leave a crease which acts as a line. Another point in our paper will exist if it lies at the intersection of two formed creases.

Definition 4.1. A point (x, y) is **origami-constructible** if, starting with our infinitely large paper with $(0, 0)$ and $(1, 0)$ marked, we can make a series of folds so that two lines intersect at (x, y). Also, if the image of an origami constructible point P after getting reflected over a constructed line l is P', then P' is an origami constructible point.

With a set of known origami-constructible points, we can create more origami constructible points. For instance, suppose $(a, 0)$ and $(0, b)$ are both origami constructible points. To show that (a, b) is an origami constructible point, we can make a fold parallel to the x-axis going through $(0, b)$, and then make a fold parallel to the y-axis going through $(0, b)$. The intersection of these two folds will be at (a, b).

Assume that every fold you make gets unfolded right afterwards (but the crease from that fold still remains).

Question 4.1. (3 pts) Suppose that (a, b) is an origami-constructible point. Explain how $(-a, -b)$ is an origami constructible point by detailing a series of folds that leads to $(-a, -b)$ being constructed.

Question 4.2. (3 pts) Suppose that (a, b) and (c, d) are both origami-constructible points. Explain how $(a + c, b + d)$ is an origami-constructible point by detailing a series of folds that leads to $(a + c, b + d)$ being constructed. You may assume that the origin and these two points are **not** collinear.

So far, we've seen how one can find more origami-constructible points from previously discovered ones through addition and additive inverses. The same can be done through the use of multiplication and multiplicative inverses.

Question 4.3. (3 pts) Suppose for $a \neq 0$ that we have the origami-constructible point $(a, 0)$. Explain how $(0, 1/a)$ is an origami-constructible point by detailing a series of folds that creates $(1, 1/a)$.

Question 4.4. (3 pts) Suppose we have the origami-constructible points $(0, a)$ and $(b, 0)$. Explain how (b, ab) is an origami-constructible point by detailing a series of folds that creates (b, ab).

Instead of thinking of our points as being in $\mathbb{R}^2$ space, we can also embed our origami-constructible points into the complex plane. With all of these addition and multiplication rules being satisfied, we can now state that the set of all origami-constructible points $\mathscr{O}$ is a **subfield** of $\mathbb{C}$, the set of all complex numbers $a + bi$, where $a, b \in \mathbb{R}$, and $i = \sqrt{-1}$.
Roughly speaking, this means that with the binary operators of adding and multiplying, the set of all origami-constructible points satisfies certain properties, including but not limited to:

- Closure: adding any two origami numbers and multiplying any two origami numbers both result in an origami number.
- Distributivity: If $a, b, c \in \mathscr{O}$, then $a * (b + c) = a * b + a * c$.
- Existence of identity elements: There exists some $e_1 \in \mathscr{O}$ where $b + e_1 = b$ for any $b \in \mathscr{O}$. Also, there exists some e_2 where $b * e_2 = b$ for any $b \in \mathscr{O}$.
- Existence of inverse elements: There exists some $i_1 \in \mathscr{O}$ where $b + i_1 = 0$ for any $b \in \mathscr{O}$. Also, there exists some i_2 where $b * i_2 = 1$ for any nonzero $b \in \mathscr{O}$.

This gives our set of origami numbers some added structure, allowing us to make more generalized claims about our set. Some other examples of fields are the rational numbers $\mathbb{Q}$ (the set of all numbers $\frac{a}{b}$, where a and b are integers and b is not equal to 0), and the complex numbers (the set of all numbers $a + bi$, where a and b are both real numbers, and $i = \sqrt{-1}$).

This definition of a field also allows us to define an order on fields.

Definition 4.2. A **field extension** K/F occurs when we have two fields, say F and K, where $F \subset K$ (meaning all of F is contained within K), K contains F as a subfield.

Some examples of this are $\mathbb{R}/\mathbb{Q}$, where $\mathbb{R}$ is the set of real numbers and $\mathbb{Q}$ is the set of rational numbers, and $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$, where $\mathbb{Q}(\sqrt{2})$ is the set of all numbers $a + b\sqrt{2}$, where a and b are rationals.

We can interpret $K = F(\alpha)$ as K being the smallest field that contains both the field F and the element α.

Definition 4.3. The **minimal polynomial** of α, is the unique monic polynomial $f(x)$ with coefficients in F of smallest degree such that $f(\alpha) = 0$.

For example, suppose $F = \mathbb{Q}$ and $\alpha = \sqrt{2}$. The minimal polynomial of $\alpha = \sqrt{2}$ over the rationals is $f(x) = x^2 - 2$, because $\sqrt{2}$ satisfies this polynomial, and $x^2 - 2$ is irreducible over the rationals (we cannot factor this polynomial into $(x - \sqrt{2})(x + \sqrt{2})$ because $\sqrt{2}$ is not rational).

Question 4.5. (6 pts) Find the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over the rational numbers $\mathbb{Q}$, and prove that this polynomial is indeed minimal.

Lemma 4.4. If p is a prime number and ω is a root of $x^p - 1 = 0$, then the minimal polynomial of ω is $x^{p-1} + x^{p-2} + \cdots + 1$.

We also define a bit more related machinery.

Definition 4.5. The **degree** of the field extension $K = F(\alpha)$ of F, denoted as $[K : F]$, is the degree of a minimal polynomial f with coefficients in F and $f(\alpha) = 0$.

Definition 4.6. A **2-3 tower** is a nested sequence of fields

$$\mathbb{Q} = F_0 \subset \cdots \subset F_n \subset \mathbb{C}$$

such that $[F_i : F_{i-1}] = 2$ or 3 for all $1 \leq i \leq n$.

We begin with our given points $(0, 0)$ and $(1, 0)$ (and our given line, the horizontal axis).
We have shown that we can use our origami folding operations to construct any point in the complex plane with rational coordinates ($a + bi$ for all $a, b \in \mathbb{Q}$, and $i = \sqrt{-1}$). This can be defined as the field extension $\mathbb{Q}(i)$.

With only this extension, our "tower" is $\mathbb{Q} \subset \mathbb{Q}(i)$, and $[\mathbb{Q}(i) : \mathbb{Q}] = 2$ because $x^2 + 1$ is the minimal polynomial of i over $\mathbb{Q}$.
Constructing $\sqrt[3]{2}$ yields us the 2-3 tower of $\mathbb{Q} \subset \mathbb{Q}(i) \subset \mathbb{Q}(i, \sqrt[3]{2})$, all of which are a subset of $\mathbb{O}$.

Question 4.6. (10 pts) Prove that if there exists a 2-3 tower

$$\mathbb{Q} = F_0 \subset \cdots \subset F_n \subset \mathbb{C}$$

such that $\alpha \in F_n$, then $\alpha \in \mathscr{O}$.

Finally, we extend this idea of foldability beyond origami-constructible points, and into the realm of

n-gons. With the information we went over, we can make a bold claim about which n-gons can be constructed by origami. However, we present some terminology so that we're better equipped for our final big theorem.

Definition 4.7. A **splitting field** of a polynomial $f(x)$ over a field F is the smallest field extension K of F in which the polynomial splits completely into linear factors. In other words, it is the smallest field containing F and all the roots of $f(x)$.

Theorem 4.8. Let $\alpha \in \mathbb{C}$ be a solution to a polynomial with coefficients in $\mathbb{Q}$, and let L be the splitting field of the minimal polynomial of α over $\mathbb{Q}$. Then, α is an origami number if and only if $[L : \mathbb{Q}] = 2^a 3^b$ for integers $a, b \geq 0$.

Definition 4.9. A prime p is called a **Pierpont prime** if $p > 3$ and p is of the form $2^a 3^b + 1$ for integers $a, b \geq 0$.

Question 4.7. (20 pts) Prove that a regular n-gon can be constructed by origami if and only if $n = 2^a 3^b p_1 p_2 \dots p_k$ for some integers $a, b \geq 0$ and where $p_1, \dots, p_k$ are distinct Pierpont primes.

As a result of this theorem, we see that the undecagon (11-gon) is the smallest regular polygon that is not constructible by straight-crease, single-fold origami!